

MULTIPLIER HOPF MONOIDS

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ABSTRACT. The notion of *multiplier Hopf monoid* in any braided monoidal category is introduced as a multiplier bimonoid whose constituent fusion morphisms are isomorphisms. In the category of vector spaces over the complex numbers, Van Daele’s definition of multiplier Hopf algebra is re-obtained. It is shown that the key features of multiplier Hopf algebras (over fields) remain valid in this more general context. Namely, for a multiplier Hopf monoid A , the existence of a unique antipode is proved — in an appropriate, multiplier-valued sense — which is shown to be a morphism of multiplier bimonoids from a twisted version of A to A . For a *regular* multiplier Hopf monoid (whose twisted versions are multiplier Hopf monoids as well) the antipode is proved to factorize through a proper automorphism of the object A . Under mild further assumptions, duals in the base category are shown to lift to the monoidal categories of modules and of comodules over a regular multiplier Hopf monoid. Finally, the so-called Fundamental Theorem of Hopf modules is proved — which states an equivalence between the base category and the category of Hopf modules over a multiplier Hopf monoid.

1. INTRODUCTION

For an arbitrary field k , the vector space of k -valued functions on a *finite* group carries a natural *Hopf algebra* structure. For not necessarily finite groups, one considers the vector space of functions of *finite support*. Its structure is reminiscent of that of a Hopf algebra but there are also some important differences; for example, the (pointwise) multiplication does not admit a unit (the constant function with value one does not have finite support). To give an algebraic description of this situation, Van Daele introduced in [16] the notion of *multiplier Hopf algebra*.

Classically, a Hopf algebra is defined as a particular type of *bialgebra* with some additional properties which are, in turn, equivalent to the existence of a so-called *antipode map*. Van Daele’s definition of multiplier Hopf algebra, however, has no version without antipode — *multiplier bialgebras* do not occur in his works. This notion was introduced later in [2] (where it appeared together with its ‘weak’ generalization). Some different structures under the same name were discussed in [9] and [15].

For many applications, it is necessary to consider bialgebras and Hopf algebras (as well as various related structures) internal to a braided monoidal category; the classical case of bialgebras or Hopf algebras over a field corresponds to working in the symmetric monoidal category of vector spaces over that field.

In [3] we initiated a program aiming to develop the theory of multiplier bialgebras and multiplier Hopf algebras in braided monoidal categories. We only considered multiplier bialgebras in [3]; in this paper we turn to multiplier Hopf algebras, still in the context of braided monoidal categories, although sometimes we need to assume that the braiding is a symmetry. The resulting analysis applies to the classical case

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of vector spaces, and also to some other cases of interest: to the symmetric monoidal categories of group-graded vector spaces, modules over a commutative ring, complex Hilbert spaces and continuous linear maps, and complete bornological vector spaces; in the case of the last two examples we rely heavily on [1].

First we study the algebraic properties of these multiplier Hopf algebras. Next we turn to duality in their monoidal categories of modules and comodules. In the particular category of vector spaces, modules over multiplier Hopf algebras appeared in [7] and comodules in [17, 12]. Finally we prove a Fundamental Theorem of Hopf Modules over multiplier Hopf algebras, stating an equivalence between the category of Hopf modules and the base category. This generalizes a classical result on Hopf algebras over fields in [13]. A study of (relative) Hopf modules for multiplier Hopf algebras over fields can be found in [18, 6].

Working in a *closed* braided monoidal category, the *multiplier monoid* of a (nice enough) semigroup can be defined: see [4]. It generalizes the construction in [5] of multiplier algebras of non-unital algebras over fields. For a multiplier Hopf monoid A in a closed braided monoidal category, the antipode can be seen as a morphism from the opposite of A to the multiplier monoid of A . In more general braided monoidal categories such a simple interpretation is not available. Still, since many of the expected properties of multiplier Hopf monoids can be proved at this higher level of generality, we prefer to work in this setting whenever it is possible.

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2. BACKGROUND

2.1. The setting. In our earlier papers [3] and [4] we worked in a braided monoidal category \mathcal{C} . In this paper we shall often need to suppose that it is symmetric; this will always be explicitly stated. The composite of morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ will be denoted by $g \circ f: A \rightarrow C$. Identity morphisms will be denoted by 1 . The monoidal product will be denoted by juxtaposition, the monoidal unit by I and the braiding by c . For n copies of the same object A , we also use the power notation $AA \dots A = A^n$. We denote by $\overline{\mathcal{C}}$ the same monoidal category \mathcal{C} with the inverse braiding $c_{Y,X}^{-1}: XY \rightarrow YX$; of course in the symmetric case this has no effect, and so $\overline{\mathcal{C}}$ is just \mathcal{C} .

As in [4], we suppose given a class \mathcal{Q} of regular epimorphisms in \mathcal{C} which is closed under composition and monoidal product, contains the isomorphisms, and is right-cancellative in the sense that if $s: A \rightarrow B$ and $t \circ s: A \rightarrow C$ are in \mathcal{Q} , then so is $t: B \rightarrow C$. Since each $q \in \mathcal{Q}$ is a regular epimorphism, it is the coequalizer of some pair of morphisms. Finally we suppose that this pair may be chosen in such a way that the coequalizer is preserved by taking the monoidal product with a fixed object. The main examples of interest are where \mathcal{Q} consists of the split epimorphisms and where \mathcal{Q} consists of the regular epimorphisms. In the latter case, we need to suppose

that the regular epimorphisms are closed under composition and under the monoidal product.

2.2. Examples. We quote from [1] some interesting examples of categories in which the assumptions of Section 2.1 hold.

Coequalizers exist in any abelian category. Since in this case any epimorphism is regular, the class of regular epimorphisms is closed under composition. All coequalizers are preserved by the monoidal product whenever the category is also closed monoidal. This includes the example of the category of G -graded k -modules, for any group G and any commutative ring k ; in particular, when G is trivial this reduces to (ungraded) k -modules, and when k is a field, to vector spaces.

Although the closed symmetric monoidal category of complete bornological vector spaces [8, 14, 19] is not abelian, it still has coequalizers, and composites of regular epimorphisms are regular epimorphisms.

The symmetric monoidal category of complex Hilbert spaces and continuous linear maps [11] is neither closed nor abelian. It is nonetheless the case that it has coequalizers, and these are preserved by the monoidal product, while regular epimorphisms are closed under composition; the latter fact follows immediately from the fact that all regular epimorphisms are split.

2.3. String diagrams. For some proofs we find it convenient to use the string diagrams of [10]. These are composed down the page, with monoidal products being taken from left to right. The braiding c and its inverse are represented by crossings, as below; the fact that they are mutually inverse allows calculations as on the right.

$$\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} | \\ | \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}$$

If an object V has a left dual \bar{V} , we denote the unit $\eta: I \rightarrow V\bar{V}$ and counit $\epsilon: \bar{V}V \rightarrow I$ of the duality using a “cap” and “cup” respectively, as in the following diagram; the triangle equations for the duality say that we may “straighten” strings, as in the equations to the right.

$$\begin{array}{c} \cap \\ \eta \end{array} \quad \begin{array}{c} \cup \\ \epsilon \end{array} \quad \begin{array}{c} \cap \\ \cup \end{array} = \begin{array}{c} | \\ | \end{array} \quad \begin{array}{c} \cup \\ \cap \end{array} = \begin{array}{c} | \\ | \end{array}$$

When reasoning about an associative multiplication, it will often be represented by a joining of two strings, as on the left below, while the counit for some comultiplication — or more generally, for a fusion morphism — will be represented as on the right.

$$\begin{array}{c} \cup \\ \downarrow \end{array}$$

2.4. Semigroups with non-degenerate multiplication. By a *semigroup* in a monoidal category \mathcal{C} we mean a pair (A, m) consisting of an object A of \mathcal{C} and a morphism $m: A^2 \rightarrow A$ — called the *multiplication* — which obeys the associativity condition $m \cdot m1 = m \cdot 1m$. If m possesses a *unit* — that is, a morphism $u: I \rightarrow A$ such that $m \cdot u1 = 1 = m \cdot 1u$ — we say that (A, m, u) is a *monoid*.

The multiplication – or, alternatively, the semigroup A – is said to be *non-degenerate* if for any objects X, Y of \mathcal{C} , both maps

$$\begin{aligned} \mathcal{C}(X, YA) &\rightarrow \mathcal{C}(XA, YA), & f &\mapsto XA \xrightarrow{f^1} YA^2 \xrightarrow{1m} YA & \text{ and} \\ \mathcal{C}(X, AY) &\rightarrow \mathcal{C}(AX, AY), & g &\mapsto AX \xrightarrow{1g} A^2Y \xrightarrow{m1} AY \end{aligned}$$

are injective. Evidently, any monoid is non-degenerate.

2.5. Examples. Our requirement in Section 2.4 that these maps be injective for *any* object Y is quite strong and can often be avoided. A more nuanced analysis of non-degeneracy is carried out in the forthcoming [1]. It is shown there that for most purposes it is enough if the injectivity condition in Section 2.4 holds for any object X , and any Y belonging to a given class. Possible choices of this class in the examples of Section 2.2 are also investigated in [1], and we now briefly recall the results of that investigation.

In the category of (group graded) vector spaces over a given field, and also in the category of complex Hilbert spaces and continuous linear maps [11], the condition of Section 2.4 holds for any object Y whenever it holds for Y being the base field.

In the category of modules over a commutative ring k , the condition of Section 2.4 holds for all locally projective [20] modules Y provided that it holds for the regular module $Y = k$.

In the category of complete bornological vector spaces [8, 14, 19], the condition of Section 2.4 holds for all complete bornological vector spaces Y with the so-called approximation property if it holds for Y taken to be the base field.

2.6. \mathbb{M} -morphisms. If (B, m) is a non-degenerate semigroup and A is any object, we define an \mathbb{M} -morphism from A to B to consist of morphisms $f_1: AB \rightarrow B$ and $f_2: BA \rightarrow B$ in \mathcal{C} for which the first diagram below

$$\begin{array}{ccccc} BAB & \xrightarrow{1f_1} & B^2 & & AB^2 & \xrightarrow{1m} & AB & & B^2A & \xrightarrow{1f_2} & B^2 \\ f_2 1 \downarrow & & \downarrow m & & f_1 1 \downarrow & & \downarrow f_1 & & m 1 \downarrow & & \downarrow m \\ B^2 & \xrightarrow{m} & B & & B^2 & \xrightarrow{m} & B & & BA & \xrightarrow{f_2} & B \end{array} \quad (2.1)$$

commutes; it follows by [4, Lemma 2.8] that the other two diagrams also commute. Given such an \mathbb{M} -morphism, we write $f: A \rightharpoonup B$, and call f_1 and f_2 the *components* of f . Using the non-degeneracy of B , \mathbb{M} -morphisms $A \rightharpoonup B$ are seen to be equal if either their first components coincide or their second components coincide.

If the braided monoidal category \mathcal{C} is *closed* (meaning that for any object X the functor $X(-) : \mathcal{C} \rightarrow \mathcal{C}$ is left adjoint) and it *has pullbacks*, then one can define the *multiplier monoid* $\mathbb{M}(B)$ of any non-degenerate semigroup B . This is a universal object characterized by the property that \mathbb{M} -morphisms $A \rightharpoonup B$ correspond bijectively to morphisms $A \rightarrow \mathbb{M}(B)$ in \mathcal{C} : see [4, Section 2.9]. For more general braided monoidal categories \mathcal{C} , no interpretation of \mathbb{M} -morphisms as morphisms in \mathcal{C} is possible.

An \mathbb{M} -morphism is said to be *dense* if its components lie in \mathcal{Q} . If the domain A is also a semigroup, the \mathbb{M} -morphism $f: A \rightharpoonup B$ is *multiplicative* if either of the

diagrams

$$\begin{array}{ccc}
 A^2B & \xrightarrow{1f_1} & AB \\
 m1 \downarrow & & \downarrow f_1 \\
 AB & \xrightarrow{f_1} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 BA^2 & \xrightarrow{1m} & BA \\
 f_21 \downarrow & & \downarrow f_2 \\
 BA & \xrightarrow{f_2} & B
 \end{array}
 \quad (2.2)$$

commutes; it then follows by [4, Remark 2.7] that both do. (If the multiplier monoid $\mathbb{M}(B)$ exists, then this is equivalent to the multiplicativity of the corresponding morphism $A \rightarrow \mathbb{M}(B)$ of \mathcal{C} in the literal sense.)

If $f: A \rightharpoonup B$ is an \mathbb{M} -morphism and $g: B \rightharpoonup C$ a dense multiplicative \mathbb{M} -morphism, there is an induced \mathbb{M} -morphism $g \bullet f: A \rightharpoonup C$ with components determined by commutativity of the following diagrams

$$\begin{array}{ccc}
 ABC & \xrightarrow{1g_1} & AC \\
 f_11 \downarrow & & \downarrow (g \bullet f)_1 \\
 BC & \xrightarrow{g_1} & C
 \end{array}
 \quad
 \begin{array}{ccc}
 CA & \xleftarrow{g_21} & CBA \\
 (g \bullet f)_2 \downarrow & & \downarrow 1f_2 \\
 C & \xleftarrow{g_2} & CB
 \end{array}
 \quad (2.3)$$

by virtue of the fact that $1g_1$ and g_21 are (regular) epimorphisms. If f is also dense and multiplicative, then so is $g \bullet f$: see [4, Proposition 3.2].

If f , g , and h are composable \mathbb{M} -morphisms, with g and h dense and multiplicative, then the associativity condition $(h \bullet g) \bullet f = h \bullet (g \bullet f)$ holds: see [4, Proposition 3.3]. If the semigroup B is not just non-degenerate but has multiplication in \mathcal{Q} , then there is a dense multiplicative \mathbb{M} -morphism $i: B \rightharpoonup B$ with both components equal to m . By [4, Proposition 3.4], this acts as a two-sided identity for the composition \bullet .

In particular, there is a category \mathcal{M} whose objects are the non-degenerate semigroups with multiplication in \mathcal{Q} , and whose morphisms are the dense multiplicative \mathbb{M} -morphisms.

If $f: A \rightharpoonup B$ is an \mathbb{M} -morphism and $z: Z \rightarrow A$ a morphism in \mathcal{C} , then there is a composite $f \circ z: Z \rightharpoonup B$ with components

$$ZB \xrightarrow{z1} AB \xrightarrow{f_1} B \xleftarrow{f_2} BA \xleftarrow{1z} BZ$$

and now if $y: Y \rightarrow Z$ is a \mathcal{C} -morphism and $g: B \rightharpoonup C$ a dense multiplicative \mathbb{M} -morphism then also $(g \bullet f) \circ z = g \bullet (f \circ z)$ and $(f \circ z) \circ y = f \circ (z \circ y)$, and finally $f \circ 1 = f$.

For a non-degenerate semigroup A , a morphism $z: Z \rightarrow A$ in \mathcal{C} induces an \mathbb{M} -morphism $z^\# : Z \rightharpoonup A$ with components

$$ZA \xrightarrow{z1} A^2 \xrightarrow{m} A \xleftarrow{m} A^2 \xleftarrow{1z} AZ$$

and $f \circ z = f \bullet z^\#$ for any multiplicative and dense \mathbb{M} -morphism $f: A \rightharpoonup B$. Consequently, for \mathcal{C} -morphisms $X \xrightarrow{x} Z \xrightarrow{z} A$, where A is a non-degenerate semigroup, $z^\# \circ x = (z \circ x)^\#$. Similarly, if $f: A \rightharpoonup B$ is an \mathbb{M} -morphism and $g: B \rightarrow C$ a multiplicative isomorphism in \mathcal{C} , then $(g^\# \bullet f)_1 = g \cdot f_1 \cdot 1g^{-1}$.

Remark 2.7. Let \mathcal{C} be a braided monoidal category. Consider an arbitrary semigroup A in \mathcal{C} and a semigroup B whose multiplication is non-degenerate. Let $f: A \rightharpoonup B$ be an \mathbb{M} -morphism, and suppose that $f_2: BA \rightarrow B$ is an epimorphism which is

preserved by taking the monoidal product with any object. Then two morphisms g' and $g'' : X \rightarrow BY$ (for any objects X and Y of \mathcal{C}) are equal if and only if

$$AX \xrightarrow{1g} ABY \xrightarrow{f_1 1} BY \quad (2.4)$$

does not depend on $g \in \{g', g''\}$. Indeed, if (2.4) does not depend on g , then since

$$\begin{array}{ccccc} BAX & \xrightarrow{11g} & BABY & \xrightarrow{1f_1 1} & B^2Y \\ f_2 1 \downarrow & & f_2 11 \downarrow & (2.1) & \downarrow m_1 \\ BX & \xrightarrow{1g} & B^2Y & \xrightarrow{m_1} & BY \end{array}$$

commutes, since $f_2 1$ is an epimorphism and since $m : B^2 \rightarrow B$ is non-degenerate, it follows that $g' = g''$.

Symmetrically, if f_1 is an epimorphism which is preserved under taking the monoidal product with any object, then two morphisms h' and $h'' : X \rightarrow YB$ are equal if and only if

$$XA \xrightarrow{h1} YBA \xrightarrow{1f_2} YB$$

does not depend on $h \in \{h', h''\}$.

Monoid morphisms in the usual sense are the same as multiplicative \mathbb{M} -morphisms whose components are (necessarily split) epimorphisms:

Lemma 2.8. *Let A and B be monoids and $f : A \rightharpoonup B$ be a multiplicative \mathbb{M} -morphism. Then there is a unique multiplicative morphism $h : A \rightarrow B$ such that $f = h^\#$. Moreover, the following conditions are equivalent to each other.*

- (a) *The component $f_1 : AB \rightarrow B$ is an epimorphism.*
- (b) *The component $f_1 : AB \rightarrow B$ is a split epimorphism.*
- (c) *The component $f_2 : BA \rightarrow B$ is an epimorphism.*
- (d) *The component $f_2 : BA \rightarrow B$ is a split epimorphism.*
- (e) *h is unital; that is, a morphism of monoids.*

Proof. Denote by u the units of the monoids A and B , and put

$$h := A \xrightarrow{1u} AB \xrightarrow{f_1} B.$$

Commutativity of the diagrams

$$\begin{array}{ccc} AB & \xrightarrow{1u1} AB^2 & \xrightarrow{f_1 1} B^2 \\ & \downarrow 1m & \downarrow m \\ AB & \xrightarrow{f_1} B \end{array} \quad \begin{array}{ccc} BA & \xrightarrow{11u} BAB & \xrightarrow{1f_1} B^2 \\ & \downarrow f_2 1 & \downarrow m \\ B & \xrightarrow{1u} B^2 & \xrightarrow{m} B \end{array}$$

proves $f = h^\#$. Since the multiplication of B has a unit, h is clearly unique with this property. By symmetry it admits an equal expression $h = f_2.u1$. With the help of

the first diagram above, also

$$\begin{array}{ccccc}
 & & 1h & & \\
 A^2 & \xrightarrow{11u} & A^2B & \xrightarrow{1f_1} & AB & \xrightarrow{h1} & B^2 \\
 m \downarrow & & m1 \downarrow & & (2.2) \searrow f_1 & & m \downarrow \\
 A & \xrightarrow{1u} & AB & \xrightarrow{f_1} & B & & \\
 & & h & & & &
 \end{array}$$

is seen to commute, proving the multiplicativity of h .

Concerning the further assertions, (b) \Rightarrow (a) and (d) \Rightarrow (c) are trivial. If (e) holds then the first diagram of

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 B & & & & \\
 u1 \downarrow & \searrow u1 & & & \\
 AB & \xrightarrow{h1} & B^2 & \xrightarrow{m} & B \\
 & & f1 & &
 \end{array} &
 \begin{array}{ccccc}
 AB & \xrightarrow{u11} & A^2B & \xrightarrow{m1} & AB \\
 f1 \downarrow & & 1f1 \downarrow & & (2.2) \searrow f1 \\
 B & \xrightarrow{u1} & AB & \xrightarrow{f1} & B
 \end{array} &
 \begin{array}{ccccc}
 I & \xrightarrow{u} & B & & \\
 u \downarrow & & u1 \searrow & & \\
 A & \xrightarrow{1u} & AB & \xrightarrow{f1} & B \\
 & & h & &
 \end{array}
 \end{array}$$

commutes, proving (b). A symmetric proof yields (e) \Rightarrow (d). Finally, if (a) holds then commutativity of the second diagram above implies that its bottom row is the identity morphism (so that in particular (b) holds). It then follows that the last diagram commutes, proving (e). The implication (c) \Rightarrow (e) follows symmetrically. \square

2.9. Monoidal structure. If (A, m) and (B, m) are non-degenerate semigroups, then so is their monoidal product $(AB, mm.1c1)$: see [4, Proposition 4.1]. If A and B have multiplication in \mathcal{Q} , then so does AB .

If $f: A \rightarrowtail C$ and $g: B \rightarrowtail D$ are \mathbb{M} -morphisms, then there is an \mathbb{M} -morphism $fg: AB \rightarrow CD$ with components

$$ABCD \xrightarrow{1c1} ACBD \xrightarrow{f_1g_1} CD \xleftarrow{f_2g_2} CADB \xleftarrow{1c1} CDAB$$

and this will be dense and multiplicative if f and g are so: see [4, Proposition 4.2]. This operation is functorial with respect to the various compositions defined in Section 2.6 (see also [4, Proposition 4.3]).

For any object X and non-degenerate semigroups A and B in a braided monoidal category \mathcal{C} , consider an \mathbb{M} -morphism $f: X \rightarrowtail AB$. By the computation

$$\begin{array}{ccccccc}
 \text{Diagram 1} & \stackrel{(2.1)}{=} & \text{Diagram 2} & \stackrel{(\text{ass})}{=} & \text{Diagram 3} & \stackrel{(2.1)}{=} & \text{Diagram 4} & \stackrel{(\text{ass})}{=} & \text{Diagram 5}
 \end{array}$$

and non-degeneracy of AB , we conclude that the first component of f renders commutative the first diagram of

$$\begin{array}{ccccccc}
 XA^2B & \xrightarrow{11c^{-1}} & XABA & \xrightarrow{f_11} & ABA & \xrightarrow{1c} & A^2B & & XAB^2 & \xrightarrow{f_11} & AB^2 \\
 1m1 \downarrow & & & & & & m1 \downarrow & & 11m \downarrow & & 1m \downarrow \\
 XAB & \xrightarrow{f_1} & AB & & & & & & XAB & \xrightarrow{f_1} & AB.
 \end{array} \quad (2.5)$$

The second diagram commutes by similar considerations and there are analogous identities for the second component.

2.10. Opposites. For a more abstract approach to the calculations in this section, see Remark 2.12 below.

If (B, m) is a semigroup, there is a semigroup $(B, m.c^{-1})$ which we call B^{op} ; clearly it is non-degenerate if and only if B is so.

Proposition 2.11. *If $f: A \rightharpoonup B$ is an \mathbb{M} -morphism, there is an \mathbb{M} -morphism $f^{\text{op}}: A \rightharpoonup B^{\text{op}}$ with components f_1^{op} and f_2^{op} given by the following composites.*

$$AB \xrightarrow{c^{-1}} BA \xrightarrow{f_2} B \xleftarrow{f_1} AB \xleftarrow{c^{-1}} BA$$

This is dense if and only if f is so. If f is multiplicative, then f^{op} is a multiplicative \mathbb{M} -morphism $f^{\text{op}}: A^{\text{op}} \rightharpoonup B^{\text{op}}$. The passage from f to f^{op} is functorial.

Proof. The fact that f^{op} is an \mathbb{M} -morphism follows from commutativity of the diagram on the left below.

$$\begin{array}{ccc}
 BAB & \xrightarrow{1c^{-1}} & B^2A \xrightarrow{1f_2} B^2 \\
 \downarrow c^{-1}1 & & \downarrow c^{-1}1 \\
 & & B^2A \\
 & & \downarrow 1c^{-1} \\
 & & BAB \xrightarrow{f_21} B^2 \\
 \downarrow f_11 & & \downarrow 1f_1 \text{ (2.1)} \\
 AB^2 & \xrightarrow{1c^{-1}} AB^2 \xrightarrow{c^{-1}1} BAB & \xrightarrow{f_21} B^2 \\
 \downarrow f_11 & & \downarrow m \\
 B^2 & \xrightarrow{c^{-1}} B^2 \xrightarrow{m} B
 \end{array}
 \qquad
 \begin{array}{ccc}
 A^2B & \xrightarrow{1c^{-1}} & ABA \xrightarrow{1f_2} AB \\
 \downarrow c^{-1}1 & & \downarrow c^{-1}1 \\
 & & BA^2 \\
 & & \downarrow 1c^{-1} \\
 & & A^2B \xrightarrow{1c^{-1}} ABA \xrightarrow{c^{-1}1} BA^2 \xrightarrow{f_21} BA \\
 \downarrow m1 & & \downarrow 1m \text{ (2.2)} \\
 AB & \xrightarrow{c^{-1}} AB \xrightarrow{c^{-1}} BA & \xrightarrow{f_2} B
 \end{array}$$

The fact that f^{op} is multiplicative if f is so follows from commutativity of the diagram on the right; the corresponding fact about f^{op} being dense is obvious. Whenever f is dense, the top rows of the following diagrams are epimorphisms. Hence functoriality follows from the equality of the left-bottom paths in the commutative diagrams

$$\begin{array}{ccc}
 ACB & \xrightarrow{1f_2} & AC \\
 \downarrow c^{-1}1 & & \downarrow c^{-1} \\
 CAB & & ACB \\
 \downarrow 1c^{-1} & & \downarrow c^{-1}1 \\
 CBA & \xrightarrow{f_21} & CA \\
 \downarrow 1g_2 & & \downarrow (f \bullet g)_2^{\text{op}} \\
 CB & \xrightarrow{f_2} & C
 \end{array}
 \qquad
 \begin{array}{ccc}
 ACB & \xrightarrow{1f_2} & AC \\
 \uparrow 1c^{-1} & & \uparrow 1c^{-1} \\
 ABC & & ACB \\
 \downarrow c^{-1}1 & & \downarrow c^{-1}1 \\
 BAC & & ABC \\
 \downarrow g_21 & & \downarrow g_21 \\
 BC & \xrightarrow{c^{-1}} CB \xrightarrow{f_2} C
 \end{array}$$

□

Remark 2.12. Write \mathcal{C}^{rev} for the monoidal category with the same underlying category \mathcal{C} , but with the reverse monoidal structure, so that the monoidal product of A and B in \mathcal{C}^{rev} is BA . The inverse-braiding c^{-1} may be used to make the identity functor

on \mathcal{C} into a strong monoidal isomorphism $\mathcal{C}^{\text{rev}} \rightarrow \mathcal{C}$; such a functor sends semigroups to semigroups, and in this case sends A to A^{op} . Furthermore, the strong monoidal isomorphism $(1, c^{-1}): \mathcal{C}^{\text{rev}} \rightarrow \mathcal{C}$ is in fact braided (if we equip \mathcal{C}^{rev} with the braiding c), from which it follows that the induced functor $\mathcal{M}^{\text{rev}} \rightarrow \mathcal{M}$ is also strong monoidal, with the binary part of the strong monoidal structure being given by $(c^{-1})^\#$.

If A and B are semigroups, then the multiplication of $A^{\text{op}}B^{\text{op}}$ is given by the composite

$$(AB)^2 \xrightarrow{1c1} A^2B^2 \xrightarrow{c^{-1}c^{-1}} A^2B^2 \xrightarrow{mm} AB$$

while the multiplication of $(AB)^{\text{op}}$ is the composite

$$(AB)^2 \xrightarrow{1c^{-1}1} A^2B^2 \xrightarrow{c^{-1}c^{-1}} A^2B^2 \xrightarrow{mm} AB$$

and clearly these are in general different, but agree if the braiding is a symmetry. A similar argument works for \mathbb{M} -morphisms, giving:

Proposition 2.13. *If the braiding of \mathcal{C} is a symmetry, then for semigroups A and B we have $A^{\text{op}}B^{\text{op}} = (AB)^{\text{op}}$, and similarly for \mathbb{M} -morphisms f and g we have $f^{\text{op}}g^{\text{op}} = (fg)^{\text{op}}$.*

In the case of an arbitrary braiding, for any \mathcal{C} -morphism $z: X \rightarrow Z$ and \mathbb{M} -morphisms $f: Z \rightharpoonup A$ and $g: Y \rightharpoonup B$, the \mathcal{C} -morphism $c: (AB)^{\text{op}} \rightarrow B^{\text{op}}A^{\text{op}}$ is an isomorphism of semigroups in \mathcal{C} and the equalities

$$(f \circ z)^{\text{op}} = f^{\text{op}} \circ z \quad \text{and} \quad g^{\text{op}} f^{\text{op}} \circ c = c^\# \bullet (fg)^{\text{op}} \quad (2.6)$$

hold.

3. MULTIPLIER HOPF MONOIDS

3.1. Multiplier bimonoids. A *multiplier bimonoid* [3] in a braided monoidal category \mathcal{C} is an object A equipped with morphisms $t_1: A^2 \rightarrow A^2$, $t_2: A^2 \rightarrow A^2$ and $e: A \rightarrow I$ subject to the axioms encoded in the following commutative diagrams (see [3]).

$$\begin{array}{ccc} A^3 & \xrightarrow{1t_1} & A^3 \xrightarrow{c1} A^3 \xrightarrow{1t_1} A^3 \xrightarrow{c^{-1}1} A^3 \\ t_1 1 \downarrow & & \downarrow t_1 1 \\ A^3 & \xrightarrow{1t_1} & A^3 \end{array} \quad \begin{array}{ccc} A^2 & & \\ t_1 \downarrow & \searrow 1e & \\ A^2 & \xrightarrow{1e} & A \end{array} \quad (3.1)$$

$$\begin{array}{ccc} A^3 & \xrightarrow{t_2 1} & A^3 \xrightarrow{1c} A^3 \xrightarrow{t_2 1} A^3 \xrightarrow{1c^{-1}} A^3 \\ 1t_2 \downarrow & & \downarrow 1t_2 \\ A^3 & \xrightarrow{t_2 1} & A^3 \end{array} \quad \begin{array}{ccc} A^2 & & \\ t_2 \downarrow & \searrow e1 & \\ A^2 & \xrightarrow{e1} & A \end{array} \quad (3.2)$$

$$\begin{array}{ccc} A^3 & \xrightarrow{t_2 1} & A^3 \\ 1t_1 \downarrow & & \downarrow 1t_1 \\ A^3 & \xrightarrow{t_2 1} & A^3 \end{array} \quad \begin{array}{ccc} A^2 & \xrightarrow{t_1} & A^2 \\ t_2 \downarrow & & \downarrow e1 \\ A^2 & \xrightarrow{1e} & A \end{array} \quad (3.3)$$

The equalities encoded in the first diagrams of (3.1) and (3.2) are referred to as the *fusion equations* and the equalities expressed by the second diagrams are termed the *counit conditions*.

The common diagonal of the second diagram of (3.3) is an associative multiplication to be denoted by m . Postcomposing the equal paths around the second diagram of (3.1) (or of (3.2)) with e , we deduce the commutativity of

$$\begin{array}{ccc} A^2 & \xrightarrow{1e} & A \\ m \downarrow & & \downarrow e \\ A & \xrightarrow{e} & I. \end{array} \quad (3.4)$$

Postcomposing the equal paths around the first diagram of (3.1) with $e11$, and postcomposing the equal paths around the first diagram of (3.2) with $11e$, we obtain the so-called *short fusion equations*

$$\begin{array}{ccccc} A^3 & \xrightarrow{1t_1} & A^3 & \xrightarrow{c1} & A^3 & \xrightarrow{1t_1} & A^3 & \xrightarrow{c^{-1}1} & A^3 & & A^3 & \xrightarrow{t_21} & A^3 & \xrightarrow{1c} & A^3 & \xrightarrow{t_21} & A^3 & \xrightarrow{1c^{-1}} & A^3 \\ m1 \downarrow & & & & \downarrow m1 & & 1m \downarrow & & & & & & & & & & & & \downarrow 1m \\ A^2 & \xrightarrow{t_1} & A^2 & & A^2 & & A^2 & \xrightarrow{t_2} & A^2. \end{array} \quad (3.5)$$

Postcomposing by $1e1$ the equal paths around the first diagram of (3.1), we obtain

$$\begin{array}{ccc} A^3 & \xrightarrow{t_11} & A^3 \\ 1m \downarrow & & \downarrow 1m \\ A^2 & \xrightarrow{t_1} & A^2. \end{array} \quad (3.6)$$

Finally, postcomposing the equal paths around the first diagram of (3.3) by $1e1$, we obtain

$$\begin{array}{ccc} A^3 & \xrightarrow{t_21} & A^3 \\ 1t_1 \downarrow & & \downarrow 1m \\ A^3 & \xrightarrow{m1} & A^2. \end{array} \quad (3.7)$$

In this paper, we only consider multiplier bimonoids (A, t_1, t_2, e) with additional properties:

- *non-degeneracy of the multiplication* $m := e1.t_1 = 1e.t_2$,
- *surjectivity of the multiplication*; in the sense that $m \in \mathcal{Q}$,
- *density of the comultiplication* [4]; that is, the property that the components

$$\begin{aligned} d_1 &:= A^3 \xrightarrow{c1} A^3 \xrightarrow{1t_1} A^3 \xrightarrow{c^{-1}1} A^3 \xrightarrow{m1} A^2 \quad \text{and} \\ d_2 &:= A^3 \xrightarrow{1c} A^3 \xrightarrow{t_21} A^3 \xrightarrow{1c^{-1}} A^3 \xrightarrow{1m} A^2 \end{aligned} \quad (3.8)$$

of the multiplicative \mathbb{M} -morphism $d: A \rightharpoonup A^2$ belong to \mathcal{Q} ;

- *density of the counit*; that is, that the counit $e: A \rightarrow I$ is in \mathcal{Q} , and so can be seen as a dense multiplicative \mathbb{M} -morphism $A \rightharpoonup I$.

Under these assumptions some of the above axioms become redundant, see [3].

Multiplier bimonoids with these properties were identified in [4, Theorem 5.1] with certain comonoids in the monoidal category \mathcal{M} described in Section 2.6.

Multiplier bimonoids are related to usual bimonoids as follows.

Proposition 3.2. *For a multiplier bimonoid (A, t_1, t_2, e) , the following assertions are equivalent.*

- (a) *The multiplication $m := e1.t_1 = 1e.t_2$ admits some unit u ; and the counit e , as well as the morphisms d_1 and d_2 of (3.8), are epimorphisms.*
- (b) *There is a bimonoid A with some monoid structure (m, u) , some comultiplication h , and the given counit e , such that $t_1 = 1m.h1$ and $t_2 = m1.1h$.*

Proof. Assume that (b) holds. Then both expressions $e1.t_1$ and $1e.t_2$ are equal to the multiplication m of the monoid A , which has a unit u by definition. Being unital, the counit e is a split epimorphism. The left-bottom path of the commutative diagram

$$\begin{array}{ccccc}
 A^3 & \xrightarrow{h11} & A^4 & \xrightarrow{1c1} & A^4 \\
 1c^{-1} \downarrow & & \downarrow 11c^{-1} & & \downarrow mm \\
 A^3 & \xrightarrow{h11} & A^4 & & \\
 & \searrow t_11 & \downarrow 1m1 & & \\
 & & A^3 & \xrightarrow{1c} & A^3 \xrightarrow{m1} A^2
 \end{array}$$

is the d_1 of (3.8), proving $d = h^\#$. So it follows by Lemma 2.8 (e) \Rightarrow (a) and (c) that d_1 and d_2 are epimorphisms so that (a) holds.

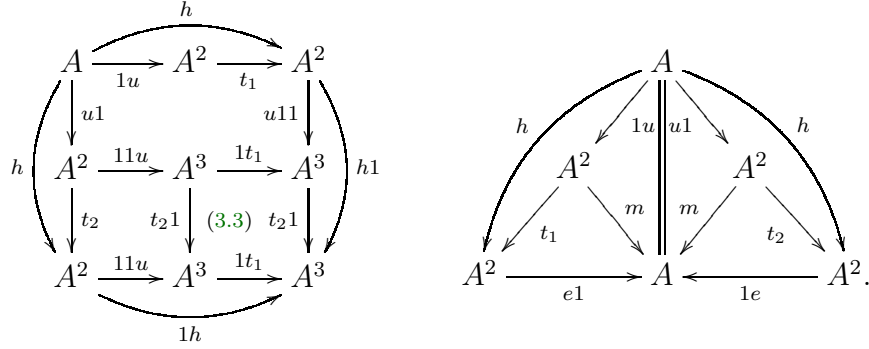
Conversely, assume that (a) holds. Then applying Lemma 2.8 (a) \Rightarrow (e) to the multiplicative \mathbb{M} -morphism $(e, e) = e^\# : A \rightarrowtail I$ we see that $e : A \rightarrow I$ is a monoid morphism. Also by Lemma 2.8 the multiplicative \mathbb{M} -morphism $d : A \rightarrowtail A^2$ is of the form $h^\#$ for the unique monoid morphism

$$h = A \xrightarrow{1uu} A^3 \xrightarrow{d_1} A^2 = A \xrightarrow{1u} A^2 \xrightarrow{t_1} A^2$$

which admits the equal form $h = t_2.u1$. It renders commutative both diagrams

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & h1 & & \\
 A^2 & \xrightarrow{u11} & A^3 & \xrightarrow{t_21} & A^3 \\
 t_1 \downarrow & & \downarrow 1t_1 & & \downarrow 1t_1 \\
 A^2 & \xrightarrow{u11} & A^3 & \xrightarrow{t_21} & A^3 \\
 & \searrow & \downarrow m1 & & \downarrow 1e1 \\
 & & A^2 & = & A^2
 \end{array}
 &
 \begin{array}{ccccc}
 & & 1h & & \\
 A^2 & \xrightarrow{11u} & A^3 & \xrightarrow{1t_1} & A^3 \\
 t_2 \downarrow & & \downarrow t_21 & & \downarrow t_21 \\
 A^2 & \xrightarrow{11u} & A^3 & \xrightarrow{1t_1} & A^3 \\
 & \searrow & \downarrow 1m & & \downarrow 1e1 \\
 & & A^2 & = & A^2
 \end{array}
 \end{array}$$

as needed for (b) to hold. Coassociativity and counitality of h follow by the commutativity of the diagrams



□

3.3. Twisting multiplier bimonoids. It is immediate from the definition that any multiplier bimonoid (A, t_1, t_2, e) determines another multiplier bimonoid $(A, c.t_2.c^{-1}, c.t_1.c^{-1}, e)$, and indeed this construction can be iterated (although if the braiding is a symmetry then further iterates do not yield anything new).

We can also understand this, in the case when the assumptions in Section 3.1 hold, in terms of the corresponding comonoids in \mathcal{M} . By the strong monoidality of the functor $(-)^{\text{op}}: \mathcal{M}^{\text{rev}} \rightarrow \mathcal{M}$, if $C = (A, d, e)$ is a comonoid in \mathcal{M} , then A^{op} also becomes a comonoid with comultiplication

$$A^{\text{op}} \xrightarrow{d^{\text{op}}} (AA)^{\text{op}} \xrightarrow{c^{\#}} A^{\text{op}} A^{\text{op}}$$

and counit $e^{\text{op}}: A^{\text{op}} \rightarrow I$. The components of the comultiplication are given by the composites

$$A^3 \xrightarrow{1c^{-1}} A^3 \xrightarrow{d_1^{\text{op}}} A^2 \xrightarrow{c} A^2 \xleftarrow{c} A^2 \xleftarrow{d_2^{\text{op}}} A^3 \xleftarrow{c^{-1}1} A^3.$$

Regard now a multiplier bimonoid A (having the properties listed in Section 3.1) as a comonoid in \mathcal{M} ; then d_1 and d_2 are defined as in (3.8). The multiplication for the twisted multiplier bimonoid is given by

$$\begin{array}{ccccc} A^2 & \xrightarrow{c^{-1}} & A^2 & \xrightarrow{t_2} & A^2 & \xrightarrow{c} & A^2 \\ & & \downarrow m & & \downarrow 1e & & \downarrow e1 \\ & & A & = & A & = & A \end{array}$$

and so the underlying semigroup is A^{op} .

The counit is just $e: A \rightarrow I$, which we can equally regard as $e^{\text{op}}: A^{\text{op}} \rightarrow I$. The first component of the comultiplication is given by the upper composite of the diagram

$$\begin{array}{ccccccc} A^3 & \xrightarrow{1c^{-1}} & A^3 & \xrightarrow{c^{-1}1} & A^3 & \xrightarrow{t_21} & A^3 & \xrightarrow{c1} & A^3 & \xrightarrow{1c} & A^3 \\ & & \downarrow d_1^{\text{op}} & & \downarrow 1c^{-1} & & \downarrow 1c^{-1} & & \downarrow c^{-1}1 & & \downarrow m1 \\ & & A^3 & & A^3 & & A^3 & & A^3 & & A^2 \\ & & \downarrow d_2 & & \downarrow 1m & & \downarrow m1 & & & & \downarrow c \\ & & A^2 & = & A^2 & = & A^2 & \xrightarrow{c} & A^2 & & \end{array}$$

which commutes by naturality of the braiding and definition of d_2 and d_1^{op} . But the lower composite is the first component of $c^\# \bullet d^{\text{op}}$ which is therefore the comultiplication.

Thus the twisted multiplier bimonoid corresponds to reversing both the multiplication and the comultiplication, as one might expect.

3.4. Morphisms between multiplier bimonoids. Let A and B be multiplier bimonoids in a braided monoidal category \mathcal{C} having the properties listed in Section 3.1. Following [4], a morphism between them is defined to be a morphism between the corresponding comonoids in the category \mathcal{M} of Section 2.6. Explicitly, this means a dense and multiplicative \mathbb{M} -morphism $f: A \rightarrowtail B$ whose components render commutative the following diagrams.

$$\begin{array}{ccc}
 \begin{array}{ccc} AB & \xrightarrow{f_1} & B \\ & \searrow ee & \downarrow e \\ & & I \end{array} & \begin{array}{ccc} A^2 B^2 & \xrightarrow{1f_1 1} & AB^2 \\ t_2 t_1 \downarrow & & \downarrow 1t_1 \\ A^2 B^2 & & AB^2 \\ 1c1 \downarrow & & \downarrow f_1 1 \\ (AB)^2 & \xrightarrow{f_1 f_1} & B^2 \end{array} & \begin{array}{ccc} BA & \xrightarrow{f_2} & B \\ & \searrow ee & \downarrow e \\ & & I \end{array} & \begin{array}{ccc} B^2 A^2 & \xrightarrow{1f_2 1} & B^2 A \\ t_2 t_1 \downarrow & & \downarrow t_2 1 \\ B^2 A^2 & & B^2 A \\ 1c1 \downarrow & & \downarrow 1f_2 \\ (BA)^2 & \xrightarrow{f_2 f_2} & B^2 \end{array}
 \end{array} \quad (3.9)$$

In fact, since the counit e of B is dense, the first and the third diagrams are equivalent to each other; and since the comultiplication of B is dense, also the second and the last diagrams are equivalent to each other; see [4, Section 6.2].

3.5. Regular multiplier bimonoids. A *regular multiplier bimonoid* [3] in a braided monoidal category \mathcal{C} is a tuple $(A, t_1, t_2, t_3, t_4, e)$ such that (A, t_1, t_2, e) is a multiplier bimonoid in \mathcal{C} and (A, t_3, t_4, e) is a multiplier bimonoid in $\bar{\mathcal{C}}$ such that the following diagrams commute, where m stands for the common diagonal of the first diagram.

$$\begin{array}{ccccc}
 \begin{array}{ccc} A^2 & \xrightarrow{t_1} & A^2 \\ \downarrow c & & \downarrow e1 \\ A^2 & & A \\ \downarrow t_3 & & \downarrow e1 \\ A^2 & \xrightarrow{e1} & A \end{array} & \begin{array}{ccc} A^3 & \xrightarrow{1t_1} & A^3 \\ \downarrow c1 & & \downarrow c1 \\ A^3 & & A^3 \\ \downarrow t_3 1 & & \downarrow 1m \\ A^3 & \xrightarrow{1m} & A^2 \end{array} & \begin{array}{ccc} A^3 & \xrightarrow{1t_1} & A^3 \\ \downarrow t_4 1 & & \downarrow t_4 1 \\ A^3 & & A^3 \\ \downarrow 1t_4 & & \downarrow m1 \\ A^3 & \xrightarrow{m1} & A^2 \end{array} & \begin{array}{ccc} A^3 & \xrightarrow{t_2 1} & A^3 \\ \downarrow 1c & & \downarrow 1c \\ A^3 & & A^3 \\ \downarrow 1t_4 & & \downarrow m1 \\ A^3 & \xrightarrow{m1} & A^2 \end{array} & \begin{array}{ccc} A^3 & \xrightarrow{t_2 1} & A^3 \\ \downarrow 1t_3 & & \downarrow 1t_3 \\ A^3 & & A^3 \\ \downarrow 1t_3 & & \downarrow 1t_3 \\ A^3 & \xrightarrow{t_2 1} & A^3 \end{array}
 \end{array} \quad (3.10)$$

Note that the first and the last diagrams, and the first and the third diagrams, respectively, imply the commutativity of

$$\begin{array}{ccc}
 A^3 & \xrightarrow{t_2 1} & A^3 \xrightarrow{1c^{-1}} A^3 \\ 1t_3 \downarrow & & \downarrow 1m \\ A^3 & \xrightarrow{m1} & A^2
 \end{array}
 \quad
 \begin{array}{ccc}
 A^3 & \xrightarrow{1t_1} & A^3 \xrightarrow{c^{-1} 1} A^3 \\ t_4 1 \downarrow & & \downarrow m1 \\ A^3 & \xrightarrow{1m} & A^2,
 \end{array} \quad (3.11)$$

from which it follows that if the multiplication is non-degenerate, there can be at most one choice of morphisms t_3 and t_4 making it into a regular multiplier bimonoid; thus for a multiplier bimonoid with non-degenerate multiplication, being regular is

a property rather than further structure. Applying (3.6) to the multiplier bimonoid (A, t_3, t_4, e) in $\bar{\mathcal{C}}$, we deduce the commutativity of

$$\begin{array}{ccccc} A^3 & \xrightarrow{1c^{-1}} & A^3 & \xrightarrow{1m} & A^2 \\ t_3 1 \downarrow & & & & \downarrow t_3 \\ A^3 & \xrightarrow{1c^{-1}} & A^3 & \xrightarrow{1m} & A^2 \end{array} \quad (3.12)$$

The multiplication m of the multiplier bimonoid (A, t_1, t_2, e) in \mathcal{C} is non-degenerate if and only if the multiplication $m.c^{-1}$ of the multiplier bimonoid (A, t_3, t_4, e) in $\bar{\mathcal{C}}$ is non-degenerate. In this case many of the above axioms become redundant: see [3, Proposition 3.11].

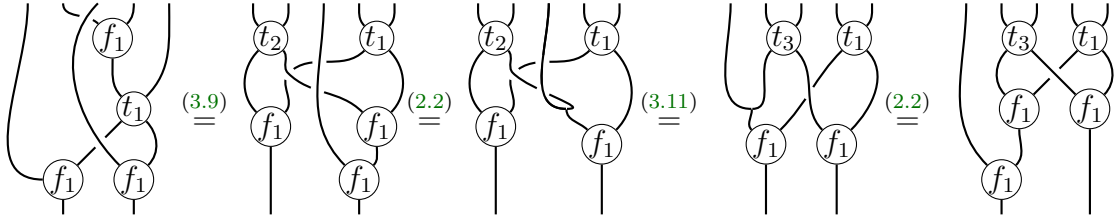
The comultiplication of the multiplier bimonoid (A, t_3, t_4, e) in $\bar{\mathcal{C}}$ is obtained from the comultiplication d of the multiplier bimonoid (A, t_1, t_2, e) in \mathcal{C} as d^{op} ; hence it is dense if and only if d is dense, and hence one multiplier bimonoid will satisfy the conditions of Section 3.1 if and only if the other does so.

Various possible reformulations of the definition of morphism of multiplier bimonoid are possible in the regular context; we shall need the following result.

Proposition 3.6. *Let $(A, t_1, t_2, t_3, t_4, e)$ and $(B, t_1, t_2, t_3, t_4, e)$ be regular multiplier bimonoids having the properties listed in Section 3.1. If f is a morphism of multiplier bimonoids from (A, t_1, t_2, e) to (B, t_1, t_2, e) , then the following diagram commutes.*

$$\begin{array}{ccccccc} A^2 & B^2 & \xrightarrow{t_3 t_1} & A^2 B^2 & \xrightarrow{1c1} & (AB)^2 & \xrightarrow{f_1 f_1} & B^2 \\ c^{-1} 11 \downarrow & & & & & & & \uparrow 1f_1 \\ A^2 B^2 & \xrightarrow{1f_1 1} & AB^2 & \xrightarrow{1t_1} & AB^2 & \xrightarrow{c1} & BAB \end{array} \quad (3.13)$$

Proof. We prove this using string diagrams, as in Section 2.3. Then the claim follows by the calculation



and Remark 2.7. □

3.7. Multiplier Hopf monoids.

Definition 3.8. A *multiplier Hopf monoid* in a braided monoidal category \mathcal{C} is a multiplier bimonoid (A, t_1, t_2, e) for which t_1 and t_2 are isomorphisms in \mathcal{C} .

Van Daele's notion of multiplier Hopf algebra [16] is equivalent to a multiplier Hopf monoid in the category of vector spaces over the complex numbers, whose multiplication is non-degenerate: see [16, Section 3].

Theorem 3.9. *Let \mathcal{C} be a braided monoidal category satisfying the standing assumptions of Section 2.1. Let (A, t_1, t_2, e) be a multiplier bimonoid in \mathcal{C} having the properties listed in Section 3.1. Then (A, t_1, t_2, e) is a multiplier Hopf monoid if and only*

if there is an \mathbb{M} -morphism $s: A \rightharpoonup A$ whose components render commutative the diagrams

$$(a) \begin{array}{ccc} A^3 & \xrightarrow{t_2 1} & A^3 \\ d_1 \downarrow & & \downarrow 1s_1 \\ A^2 & \xleftarrow{t_1} & A^2 \end{array} \quad (b) \begin{array}{ccc} A^3 & \xrightarrow{1t_1} & A^3 \\ d_2 \downarrow & & \downarrow s_2 1 \\ A^2 & \xleftarrow{t_2} & A^2. \end{array}$$

Such an \mathbb{M} -morphism is unique, dense, and is called the antipode of the multiplier Hopf monoid.

Proof. If t_1 and t_2 are invertible, define the components of s to be

$$s_1 := A^2 \xrightarrow{t_1^{-1}} A^2 \xrightarrow{e1} A \quad \text{and} \quad s_2 := A^2 \xrightarrow{t_2^{-1}} A^2 \xrightarrow{1e} A. \quad (3.14)$$

Commutativity of the first diagram in

$$(3.15) \quad \begin{array}{ccc} \begin{array}{c} A^3 \xrightarrow{1t_1} A^3 \\ \downarrow t_2^{-1} 1 \quad (3.3) \\ A^3 \xrightarrow{1t_1} A^3 \\ \downarrow t_2 1 \quad m1 \\ A^3 \xrightarrow{1e1} A^2 \end{array} & \begin{array}{c} A^3 \xrightarrow{1t_1} A^3 \\ \downarrow t_2^{-1} 1 \\ A^3 \xrightarrow{1e1} A^2 \\ \downarrow m \\ A^2 \end{array} & \begin{array}{c} A^3 \xrightarrow{1t_1} A^3 \\ \downarrow t_2^{-1} 1 \\ A^3 \xrightarrow{1e1} A^2 \\ \downarrow m \\ A^2 \end{array} \\ \begin{array}{c} A^3 \xrightarrow{1e1} A^2 \\ \downarrow m \\ A \end{array} & \begin{array}{c} A^3 \xrightarrow{1t_1} A^3 \\ \downarrow t_2^{-1} 1 \\ A^3 \xrightarrow{1e1} A^2 \\ \downarrow m \\ A^2 \end{array} & \begin{array}{c} A^3 \xrightarrow{1t_1} A^3 \\ \downarrow t_2^{-1} 1 \\ A^3 \xrightarrow{1e1} A^2 \\ \downarrow m \\ A^2 \end{array} \end{array} \quad (3.15)$$

together with the invertibility of t_1 implies $m.1s_1 = m.1e1.1t_1^{-1} = m.s_2 1$ so that s_1 and s_2 define an \mathbb{M} -morphism $s: A \rightharpoonup A$. Then diagram (a) commutes by the commutativity of the second diagram of (3.15) and a symmetrical reasoning applies to diagram (b).

Suppose conversely that s is an \mathbb{M} -morphism as in the theorem. The first diagram of

$$(3.16) \quad \begin{array}{ccc} \begin{array}{c} A^3 \xrightarrow{1e1} A^2 \\ \downarrow t_2 1 \\ A^3 \xrightarrow{1s_1} A^2 \\ \downarrow s_2 1 \quad (2.1) \\ A^2 \xrightarrow{m} A \end{array} & \begin{array}{c} A^3 \xrightarrow{1e1} A^2 \\ \downarrow d_1 \\ A^3 \xrightarrow{1s_1} A^2 \\ \downarrow m \\ A^2 \end{array} & \begin{array}{c} A^3 \xrightarrow{1e1} A^2 \\ \downarrow d_1 \\ A^3 \xrightarrow{1s_1} A^2 \\ \downarrow m \\ A^2 \end{array} \\ \begin{array}{c} A^3 \xrightarrow{1e1} A^2 \\ \downarrow d_1 \\ A^3 \xrightarrow{1s_1} A^2 \\ \downarrow m \\ A^2 \end{array} & \begin{array}{c} A^3 \xrightarrow{1e1} A^2 \\ \downarrow d_1 \\ A^3 \xrightarrow{1s_1} A^2 \\ \downarrow m \\ A^2 \end{array} & \begin{array}{c} A^3 \xrightarrow{1e1} A^2 \\ \downarrow d_1 \\ A^3 \xrightarrow{1s_1} A^2 \\ \downarrow m \\ A^2 \end{array} \end{array} \quad (3.16)$$

commutes; where the region on the right commutes by the commutativity of the second diagram of (3.16). By the non-degeneracy of m this implies that the second

diagram of

$$\begin{array}{ccc}
 A^2 & \xrightarrow{t_1} & A^2 \\
 & \searrow e1 & \downarrow s_1 \\
 & & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A^2 & \xrightarrow{t_2} & A^2 \\
 & \searrow 1e & \downarrow s_2 \\
 & & A
 \end{array}
 \tag{3.17}$$

commutes, and similarly the first diagram also commutes.

Now d_1 is in \mathcal{Q} by assumption, hence by commutativity of (a), t_1 is certainly an epimorphism preserved by monoidal product, and in fact will be a regular epimorphism provided that $t_2 1$ and $1 s_1$ are epimorphisms. But $t_2 1$ is an epimorphism by commutativity of (b) and the fact that $d_2 \in \mathcal{Q}$, while $1 s_1$ will be an epimorphism by commutativity of (3.17) since $e \in \mathcal{Q}$ implies that $1 e 1$ is an epimorphism. Thus t_1 is a regular epimorphism; if we can show that it is a monomorphism then it will be invertible.

For any object X , the function $\mathcal{C}(X, A^2) \rightarrow \mathcal{C}(AX, A^2)$ which sends f to the equal paths of

$$\begin{array}{ccccc}
 AX & \xrightarrow{1f} & A^3 & \xrightarrow{1t_1} & A^3 \\
 & & \downarrow t_2 1 & \text{(3.3)} & \downarrow t_2 1 \\
 & & A^3 & \xrightarrow{1t_1} & A^3 \\
 & \searrow m1 & \downarrow 1e1 & \text{(3.17)} & \downarrow 1s_1 \\
 & & A^2 & \xlongequal{\quad} & A^2
 \end{array}$$

is injective by the non-degeneracy of m ; thus the function sending f to $t_1.f$ is also injective, and so t_1 is a monomorphism.

This proves that t_1 is invertible, while t_2 is invertible by a symmetric reasoning. Finally the uniqueness and the density of s clearly follow from the commutativity of (3.17), since $e \in \mathcal{Q}$ and both t_1 and t_2 are invertible. \square

Remark 3.10. Let \mathcal{C} be a braided monoidal category satisfying our standing assumptions of Section 2.1. Let (A, t_1, t_2, e) be a multiplier Hopf monoid in \mathcal{C} and consider the following assertions.

- (a) The counit $e : A \rightarrow I$ belongs to \mathcal{Q} .
- (b) The multiplication $m : A^2 \rightarrow A$ belongs to \mathcal{Q} .
- (c) The component $s_1 : A^2 \rightarrow A$ of the antipode belongs to \mathcal{Q} .
- (d) The component $s_2 : A^2 \rightarrow A$ of the antipode belongs to \mathcal{Q} .
- (e) The morphism $d_1 : A^3 \rightarrow A^2$ of (3.8) belongs to \mathcal{Q} .
- (f) The morphism $d_2 : A^3 \rightarrow A^2$ of (3.8) belongs to \mathcal{Q} .

Since t_1 is invertible and $s_1.t_1 = e1$, condition (c) is equivalent to $e1$ belonging to \mathcal{Q} , which follows from (a). Dually (d) is equivalent to $1e$ belonging to \mathcal{Q} , and since $1e$ is obtained from $e1$ by conjugating with the braiding, this is equivalent to (c). Since $m = e1.t_1$, it belongs to \mathcal{Q} if and only if $e1$ does, thus (b) is equivalent to (c). Since $d_1 = t_1.1s_1.t_2 1$, it belongs to \mathcal{Q} if and only if $1s_1$ does so, thus if and only if $1e1$ does so; this is in turn equivalent to d_2 belonging to \mathcal{Q} , and so (e) is equivalent to (f), and these follow from (c).

In summary, (a) \Rightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Rightarrow (e) \Leftrightarrow (f). Thus for a multiplier Hopf monoid (A, t_1, t_2, e) with non-degenerate multiplication, all of the properties listed in Section 3.1 follow from the single assumption (a).

From Lemma 2.8 and Proposition 3.2 we obtain the following.

Theorem 3.11. *Consider a braided monoidal category \mathcal{C} satisfying our standing assumptions of Section 2.1. For a multiplier Hopf monoid (A, t_1, t_2, e) in \mathcal{C} , the following assertions are equivalent.*

- (a) *The multiplication $m := e1.t_1 = 1e.t_2$ admits some unit u , and the counit e is an epimorphism.*
- (b) *There is a Hopf monoid A with some monoid structure (m, u) , some comultiplication h , and the given counit e , such that $t_1 = 1m.h1$ and $t_2 = m1.1h$.*

Proof. (b) implies (a) by Proposition 3.2.

Conversely, if (a) holds then applying Lemma 2.8 (a) \Rightarrow (b) to the multiplicative \mathbb{M} -morphism $(e, e) : A \rightharpoonup I$ we see that e is a split epimorphism. Then $1e1 : A^3 \rightarrow A^2$ is an epimorphism as well and therefore so are $1s_1 = 1e1.t_1^{-1}$ and $d_1 = t_1.1s_1.t_21$ differing from it by isomorphisms. Then also d_2 is an epimorphism by Lemma 2.8 (a) \Rightarrow (c) and thus we can apply Proposition 3.2 to conclude that A is a bimonoid whose multiplication m and comultiplication h obey $t_1 = 1m.h1$ and $t_2 = m1.1h$. Since t_1 and t_2 are isomorphisms by assumption, A is a Hopf monoid proving (b). \square

Theorem 3.12. *Consider a braided monoidal category \mathcal{C} satisfying our standing assumptions of Section 2.1. Let (A, t_1, t_2, e) be a multiplier Hopf monoid in \mathcal{C} with non-degenerate multiplication and dense counit. Then its antipode s is a morphism of multiplier bimonoids $(A, c.t_2.c^{-1}, c.t_1.c^{-1}, e) \rightarrow (A, t_1, t_2, e)$ in the sense of Paragraph 3.4.*

Proof. By Remark 3.10, also the multiplication m and the components d_1 and d_2 of the comultiplication belong to \mathcal{Q} . In light of Paragraph 3.3, it will suffice to show that s defines a morphism of comonoids in \mathcal{M} , from $A_{\text{op}}^{\text{op}}$ to A . We already know that s is a dense \mathbb{M} -morphism; it is multiplicative by commutativity of the diagram

$$\begin{array}{ccccc}
 A^3 & \xrightarrow{c^{-1}1} & A^3 & \xrightarrow{m1} & A^2 \\
 \downarrow 1t_1^{-1} & & \downarrow 1t_1^{-1} & & \downarrow t_1^{-1} \\
 A^3 & \xrightarrow{c^{-1}1} & A^3 & \xrightarrow{m1} & A^2 \\
 \downarrow 1e1 & & \downarrow e11 & & \downarrow e1 \\
 A^2 & \xrightarrow{t_1^{-1}} & A^2 & \xrightarrow{e1} & A
 \end{array}$$

$\begin{array}{c} \text{(3.5)} \\ \text{(3.4)} \end{array}$

$\begin{array}{c} 1s_1 \quad s_1 \\ s_1 \end{array}$

and so is at least a morphism in \mathcal{M} . Next we should show that it preserves the comonoid structure. Preservation of the counit follows by commutativity of the diagram

$$\begin{array}{ccccc}
 & & s_1 & & \\
 A^2 & \xrightarrow{\quad} & A^2 & \xrightarrow{\quad} & A \\
 \downarrow 1e & \scriptstyle t_1^{-1} & \downarrow 1e & \scriptstyle e1 & \downarrow e \\
 A & \xrightarrow{\quad} & A & \xrightarrow{\quad} & I \\
 & & e & &
 \end{array}
 \quad (3.1)$$

and so it remains to show that s preserves the comultiplication.

We do this using string diagrams, as in Section 2.3. Introduce the shorthand notation $f := d \bullet s : A \rightharpoonup A^2$, so that $f_1.1d_1 = d_1.s_1.11$. Then

By non-degeneracy of the multiplication the resulting equality is in turn equivalent to the equality

We may now use the inverse of t_1 and the formula $s_1 = e1.t_1^{-1}$ to write $f_1 = (d \bullet s)_1$ as the expression on the left below

which, by the fusion equation (3.1), is equal to the expression on the right. Using the formula $s_1 = e1.t_1^{-1}$ once again, we see that $(d \bullet s)_1$ is given by the composite

$$A^3 \xrightarrow{t_1^{-1}1} A^3 \xrightarrow{c1} A^3 \xrightarrow{1s_1} A^2.$$

We must show that this is equal to the first component of $ss \bullet c^\# \bullet d^{\text{op}}$; in other words, we must show that the equality

(3.18)

holds. By the definition of d_2 and routine braid calculations the left hand side of this is equal to the left hand side of the following chain of calculations.

Here the first equality holds by multiplicativity of s , the second by the defining property of (the bottom left) s_1 , the third by (3.3), and the fourth by (one) definition of the multiplication m ; finally the right hand side is equal to the right hand side of (3.18) by multiplicativity of s once again. \square

Definition 3.13. A *regular multiplier Hopf monoid* in a braided monoidal category \mathcal{C} is a regular multiplier bimonoid $(A, t_1, t_2, t_3, t_4, e)$ such that (A, t_1, t_2, e) is a multiplier Hopf monoid in \mathcal{C} and (A, t_3, t_4, e) is a multiplier Hopf monoid in $\overline{\mathcal{C}}$; that is, such that t_1, t_2, t_3 , and t_4 are all isomorphisms in \mathcal{C} .

Van Daele's notion of regular multiplier Hopf algebra [16] is equivalent to a regular multiplier Hopf monoid in the category of vector spaces over the complex numbers, whose multiplication is non-degenerate.

A Hopf monoid — regarded as a multiplier Hopf monoid — is regular if and only if the antipode is invertible.

Corollary 3.14. Let \mathcal{C} be a braided monoidal category satisfying our standing assumptions of Section 2.1. Let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier Hopf monoid in \mathcal{C} with non-degenerate multiplication and dense counit. Applying Theorem 3.9 to the multiplier Hopf monoid (A, t_3, t_4, e) in $\overline{\mathcal{C}}$, we conclude that the components

$$s'_1 := A^2 \xrightarrow{t_3^{-1}} A^2 \xrightarrow{e1} A \quad \text{and} \quad s'_2 := A^2 \xrightarrow{t_4^{-1}} A^2 \xrightarrow{1e} A \quad (3.19)$$

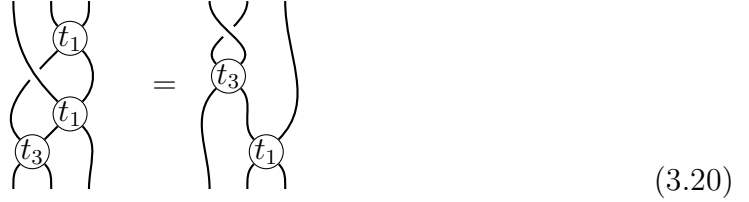
of the antipode $s': A \rightarrow A^{\text{op}}$ of (A, t_3, t_4, e) render commutative the diagrams

$$\begin{array}{ccccc} A^3 & \xrightarrow{c1} & A^3 & \xrightarrow{s'_1 1} & A^2 \\ 1c \downarrow & & & & \downarrow m \\ A^3 & \xrightarrow{1s'_2} & A^2 & \xrightarrow{m} & A \end{array} \quad \begin{array}{ccccc} A^3 & \xrightarrow{c_{A^2, A}} & A^3 & \xrightarrow{t_4 1} & A^3 \\ d_2 \downarrow & & & & \downarrow 1s'_1 \\ A^2 & \xleftarrow{t_3} & A^2 & & \end{array} \quad \begin{array}{ccccc} A^3 & \xrightarrow{c_{A, A^2}} & A^3 & \xrightarrow{1t_3} & A^3 \\ d_1 \downarrow & & & & \downarrow s'_2 1 \\ A^2 & \xleftarrow{t_4} & A^2 & & \end{array}$$

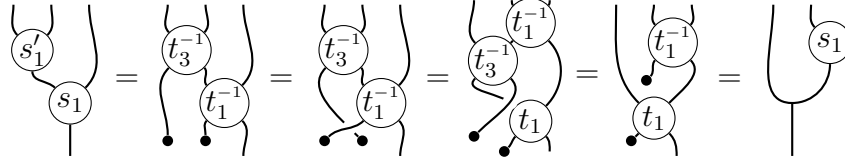
Theorem 3.15. *Let \mathcal{C} be a braided monoidal category satisfying our standing assumptions of Section 2.1. Let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier Hopf monoid in \mathcal{C} with non-degenerate multiplication and dense counit. Then the \mathbb{M} -morphisms $s: A^{\text{op}} \rightarrow A$ in Theorem 3.9 and $s': A \rightarrow A^{\text{op}}$ in Corollary 3.14 are mutually inverse in \mathcal{M} .*

Proof. We shall prove that $s \bullet s'$ is the identity \mathbb{M} -morphism; the other inverse law holds by a symmetric argument. Furthermore, by non-degeneracy, it will suffice to show that the first components agree; in other words, that $(s \bullet s')_1 = m$, or equivalently that $s_1 \cdot s'_1 1 = m \cdot 1 s_1$.

To do this we use the compatibility condition



which appeared in [3, Remark 3.10], in the following calculation.



□

Theorem 3.16. *Let \mathcal{C} be a braided monoidal category satisfying our standing assumptions of Section 2.1. For a regular multiplier Hopf monoid $(A, t_1, t_2, t_3, t_4, e)$ whose multiplication is non-degenerate and whose counit is dense, the following assertions hold.*

- (1) *There is a unique morphism $\bar{s}: A \rightarrow A$ such that $s = \bar{s}^\#$.*
- (2) *There is a unique morphism $\bar{s}': A \rightarrow A$ such that $s' = \bar{s}'^\#$.*
- (3) *The morphisms \bar{s} in part (1) and \bar{s}' in part (2) are mutually inverse isomorphisms in \mathcal{C} .*

Proof. This follows immediately from Theorem 3.15 and [4, Proposition 3.10]. □

If \mathcal{C} is the category of vector spaces over the field of complex numbers, then Theorem 3.16 reduces to [16, Proposition 5.2].

Corollary 3.17. *Applying [4, Example 6.3], we conclude from Theorem 3.12 and Theorem 3.16 that for a regular multiplier Hopf monoid $(A, t_1, t_2, t_3, t_4, e)$ whose multiplication is non-degenerate and whose counit is dense, the morphism $\bar{s}: A \rightarrow A$ in Theorem 3.16 obeys $e \cdot \bar{s} = e$ and it renders commutative the diagram*

$$\begin{array}{ccccccc}
 A^2 & \xrightarrow{c^{-1}} & A^2 & \xrightarrow{t_2} & A^2 & \xrightarrow{c} & A^2 \\
 \bar{s} \downarrow & & & & & & \downarrow \bar{s} \\
 A^2 & & \xrightarrow{t_1} & & A^2 & &
 \end{array}$$

(so that $\bar{s} \cdot m \cdot c^{-1} = m \cdot \bar{s} \bar{s}$).

4. COMODULES OVER MULTIPLIER HOPF MONOIDS

For a regular multiplier bimonoid in a braided monoidal category \mathcal{C} , the category of comodules was shown in [3] to have a monoidal structure lifting that of \mathcal{C} . The aim in this section is to show that if the underlying multiplier bimonoid is Hopf, then a comodule possesses a dual if and only if the underlying object of \mathcal{C} does so.

4.1. Comodules. A *comodule* [3] over a regular multiplier bimonoid $(A, t_1, t_2, t_3, t_4, e)$ is a tuple (V, v^1, v^3) , where $v^1 : VA \rightarrow VA$ is a comodule over t_1 [3], in the sense that it renders commutative the diagrams

$$\begin{array}{ccc} VA^2 & \xrightarrow{1t_1} & VA^2 \xrightarrow{c1} AVA \xrightarrow{1v^1} AVA \xrightarrow{c^{-1}1} VA^2 \\ v^1 1 \downarrow & & \downarrow v^1 1 \\ VA^2 & \xrightarrow{1t_1} & VA^2 \end{array} \quad \begin{array}{ccc} VA & \xrightarrow{v^1} & VA \\ & \searrow 1e & \downarrow 1e \\ & & V \end{array} \quad (4.1)$$

and $v^3 : VA \rightarrow VA$ is a comodule over t_3 [3], in the sense that it renders commutative the diagrams

$$\begin{array}{ccc} VA^2 & \xrightarrow{1t_3} & VA^2 \xrightarrow{c^{-1}1} AVA \xrightarrow{1v^3} AVA \xrightarrow{c1} VA^2 \\ v^3 1 \downarrow & & \downarrow v^3 1 \\ VA^2 & \xrightarrow{1t_3} & VA^2 \end{array} \quad \begin{array}{ccc} VA & \xrightarrow{v^3} & VA \\ & \searrow 1e & \downarrow 1e \\ & & V \end{array} \quad (4.2)$$

and where finally v_1 and v_3 satisfy the compatibility condition asserting the commutativity of

$$\begin{array}{ccc} AVA & \xrightarrow{1v^1} & AVA \xrightarrow{c1} VA^2 \\ c1 \downarrow & & \downarrow 1m \\ VA^2 & \xrightarrow{v^3 1} & VA^2 \xrightarrow{1m} VA \end{array} \quad (4.3)$$

A *morphism of comodules* $(V, v^1, v^3) \rightarrow (W, w^1, w^3)$ is a morphism $f : V \rightarrow W$ in \mathcal{C} such that the following diagrams commute.

$$\begin{array}{ccc} VA & \xrightarrow{f^1} & WA \\ v^1 \downarrow & & \downarrow w^1 \\ VA & \xrightarrow{f_1} & WA \end{array} \quad \begin{array}{ccc} VA & \xrightarrow{f^1} & WA \\ v^3 \downarrow & & \downarrow w^3 \\ VA & \xrightarrow{f_1} & WA \end{array} \quad (4.4)$$

If the multiplication is non-degenerate, then it is an easy consequence of (4.3) that the following diagrams commute.

$$\begin{array}{ccc} VA^2 & \xrightarrow{v^1 1} & VA^2 \\ 1m \downarrow & & \downarrow 1m \\ VA & \xrightarrow{v^1} & VA \end{array} \quad \begin{array}{ccc} VA^2 & \xrightarrow{c^{-1}1} & AVA \xrightarrow{1v^3} AVA \xrightarrow{c1} VA^2 \\ 1m \downarrow & & \downarrow 1m \\ VA & \xrightarrow{v^3} & VA \end{array} \quad (4.5)$$

In the second of these, the top path is equal to $1c^{-1}.v^3 1.1c$.

Lemma 4.2. *Let (A, t_1, t_2, t_3, e) be a regular multiplier bimonoid in a braided monoidal category \mathcal{C} . Assume that it satisfies the conditions listed in Section 3.1. For any morphisms $v^1 : VA \rightarrow VA$ and $v^3 : VA \rightarrow VA$ rendering commutative (4.3), the following assertions are equivalent.*

- (a) (V, v^1, v^3) is an A -comodule.
- (b) (V, v^1) is a t_1 -comodule (that is, it renders commutative the diagrams of (4.1)).
- (c) (V, v^3) is a t_3 -comodule (that is, it renders commutative the diagrams of (4.2)).

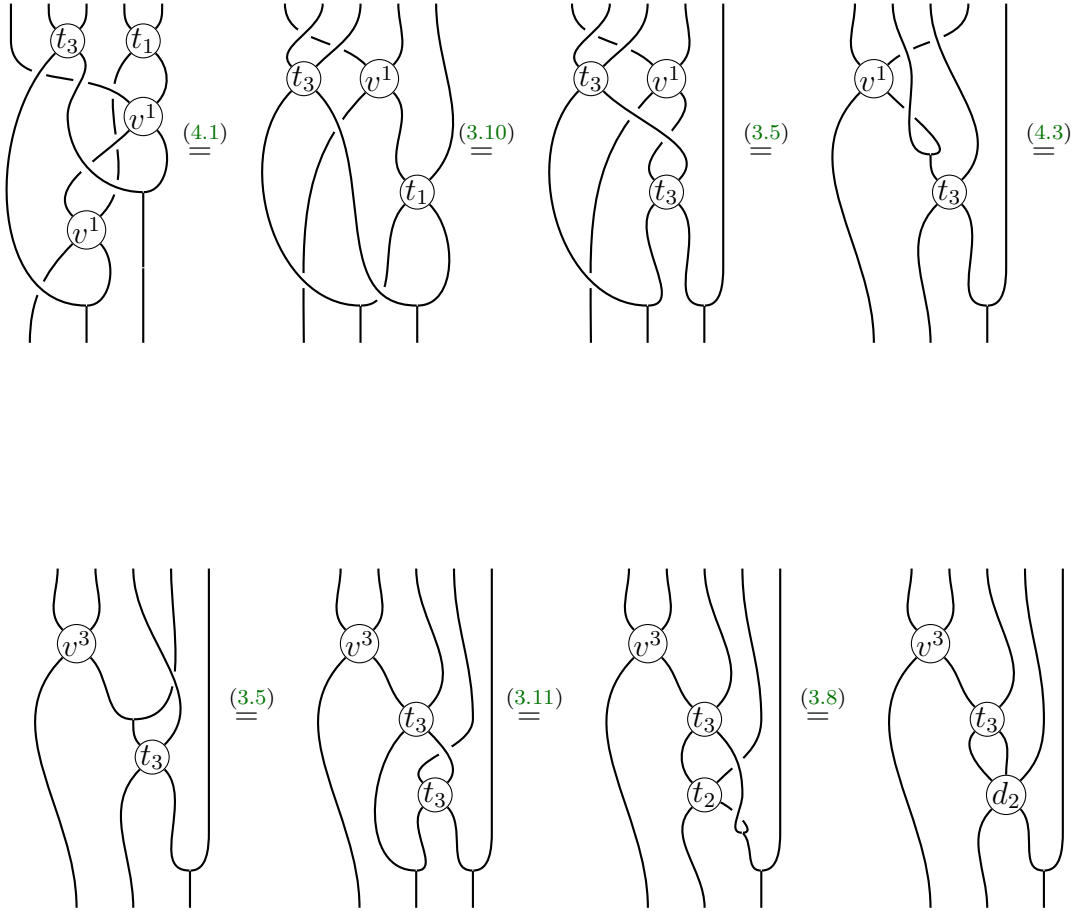
Proof. Since (a) means that (b) and (c) simultaneously hold, it suffices to prove the equivalence of (b) and (c). We show that (b) implies (c); the opposite implication follows symmetrically.

Since $11e : VA^2 \rightarrow VA$ is an epimorphism by assumption, the second diagram of (4.2) commutes by the commutativity of

$$\begin{array}{ccccc}
 VA & \xleftarrow{11e} & VA^2 & \xrightarrow{11e} & VA \\
 \downarrow c^{-1} & & \downarrow c^{-1}1 & \searrow v^3 1 & \downarrow v^3 \\
 AV & \xleftarrow{11e} & AVA & & VA \\
 \parallel & \text{(4.1)} & \downarrow 1v^1 & & \downarrow 1e \\
 AV & \xleftarrow{11e} & AVA & \xrightarrow{(4.3)} & VA^2 \xrightarrow{11e} VA \\
 \downarrow c & & \downarrow c1 & \downarrow 1m & \downarrow 1e \\
 & & VA^2 & \xrightarrow{1m} & VA \xrightarrow{(3.4)} V \\
 \swarrow 11e & & \swarrow 1e & \searrow 1e & \\
 VA & \xrightarrow{1e} & & & V
 \end{array}$$

In order to see that the first diagram of (4.2) commutes, we use the following equalities

$$\begin{array}{ccccccc}
 \text{Diagram 1} & \stackrel{(3.8)}{=} & \text{Diagram 2} & \stackrel{(\text{ass})}{=} & \text{Diagram 3} & \stackrel{(3.7)}{=} & \text{Diagram 4} \stackrel{(4.3)}{=} \\
 \text{(with } t_3, v^3, d_2 \text{)} & & \text{(with } t_3, v^3, t_2 \text{)} & & \text{(with } t_3, v^3, t_2 \text{)} & & \text{(with } t_3, v^3, t_1 \text{)}
 \end{array}$$



and then the claim follows from equality of the first and last of these, the non-degeneracy of m , and Remark 2.7. \square

Lemma 4.3. *Let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier bimonoid in a braided monoidal category \mathcal{C} having the properties listed in Section 3.1. For any A -comodules (V, v^1, v^3) , (W, w^1, w^3) and any morphism $f : V \rightarrow W$ in \mathcal{C} , the following assertions are equivalent.*

- (a) f is a morphism of comodules.
- (b) f renders commutative the first diagram of (4.4).
- (c) f renders commutative the second diagram of (4.4).

Proof. Since (a) means that (b) and (c) simultaneously hold, it suffices to prove the equivalence of (b) and (c). This follows by noting that – thanks to the non-degeneracy of m – (b) is equivalent to the commutativity of the exterior; and (c) is equivalent to

the commutativity of the inner square, in

$$\begin{array}{ccccccc}
 AVA & \xrightarrow{1v^1} & AVA & \xrightarrow{c1} & VA^2 & \xrightarrow{1m} & VA \\
 & \searrow c1 & & & \downarrow f1 & & \downarrow f1 \\
 & & VA^2 & \xrightarrow{v^3 1} & VA^2 & \xrightarrow{1m} & VA \\
 & & \downarrow f1 & & \downarrow f1 & & \\
 & & WA^2 & \xrightarrow{w^3 1} & WA^2 & \xrightarrow{1m} & WA \\
 & \nearrow c1 & & & & & \\
 AWA & \xrightarrow{1w^1} & AWA & \xrightarrow{c1} & WA^2 & \xrightarrow{1m} & WA
 \end{array}
 \quad (4.3)$$

□

4.4. Change of base for comodules. A morphism of (counital) coalgebras induces a functor between the corresponding categories of comodules. There is a corresponding construction for comodules over a regular multiplier bimonoid.

Proposition 4.5. *Let A and B be regular multiplier bimonoids having the properties listed in Section 3.1, and let $f: A \rightharpoonup B$ be a dense multiplicative \mathbb{M} -morphism.*

- (1) *If $v^1: VA \rightarrow VA$ and $v^3: VA \rightarrow VA$ satisfy (4.3), then there is a unique pair of morphisms $w^1: VB \rightarrow VB$ and $w^3: VB \rightarrow VB$ which satisfy (4.3) and make the following diagrams commute.*

$$\begin{array}{ccc}
 VAB \xrightarrow{1f_1} VB & VAB \xrightarrow{1c^{-1}} VBA \xrightarrow{1f_2} VB & \\
 v^1 1 \downarrow & \downarrow w^1 & \\
 VAB \xrightarrow{1f_1} VB & VAB \xrightarrow{1c^{-1}} VBA \xrightarrow{1f_2} VB & \\
 & v^3 1 \downarrow & \downarrow w^3
 \end{array}
 \quad (4.6)$$

- (2) *If f is a morphism of multiplier bimonoids and (V, v^1, v^3) is an A -comodule, then (V, w^1, w^3) is a B -comodule.*

The construction in part (2) gives the object map of a functor f_* acting on the morphisms as the identity map, from the category of A -comodules to the category of B -comodules. For morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ of multiplier bimonoids, the composite functor $g_* f_*$ is naturally isomorphic to $(g \bullet f)_*$ and the identity \mathbb{M} -morphism $A \rightharpoonup A$ induces the identity functor. If we regard \mathcal{M} as a bicategory with only identity 2-cells, then this defines a pseudofunctor from \mathcal{M} to the 2-category of categories, functors and natural transformations.

Proof. First we verify the existence of w^1 and w^3 satisfying the defining relations (4.6). For the case of w^1 , we use the fact that f_1 is the coequalizer of some pair of morphisms $x, x': X \rightarrow AB$, and that this coequalizer is preserved by taking the monoidal product

with V . In the diagram

$$\begin{array}{ccccccc}
 AVAB & \xlongequal{\quad} & AVAB & \xrightarrow{1v^11} & AVAB & \xrightarrow{11f_1} & AVB & \xrightarrow{1c^{-1}} & ABV \\
 \downarrow 11f_1 & & \downarrow c11 & & \downarrow c11 & & \downarrow c1 & & \downarrow f_11 \\
 & & VA^2B & \xrightarrow{(4.3)} & VA^2B & \xrightarrow{11f_1} & VAB & & \\
 & & \downarrow v^311 & & \downarrow 1m1 & & \downarrow 1f_1 & & \\
 AVB & & VA^2B & \xrightarrow{1m1} & VAB & \xrightarrow{(2.2)} & VAB & & \\
 \downarrow c1 & & \downarrow 11f_1 & & \downarrow 1f_1 & & \downarrow 1f_1 & & \\
 VAB & \xrightarrow{v^31} & VAB & \xrightarrow{1f_1} & VB & \xlongequal{\quad} & VB & \xrightarrow{c^{-1}} & BV
 \end{array}$$

the left-bottom composite coequalizes $11x$ and $11x'$, hence so does the top-right composite. Now by Remark 2.7 it follows that $c^{-1}.1f_1.v^11$ coequalizes $1x$ and $1x'$, hence so too does $1f_1.v^11$. The existence of w^1 follows. The case of w^3 is similar.

By the calculation

and density of f , we see that the diagram

$$\begin{array}{ccccc}
 AVB & \xrightarrow{1w^1} & AVB & \xrightarrow{c1} & VAB \\
 \downarrow c1 & & & & \downarrow 1f_1 \\
 VAB & \xrightarrow{v^31} & VAB & \xrightarrow{1f_1} & VB
 \end{array} \tag{4.7}$$

commutes. (In light of Remark 2.7 this actually characterizes w^1 .)

The compatibility condition (4.3) for w^1 and w^3 follows from the calculation

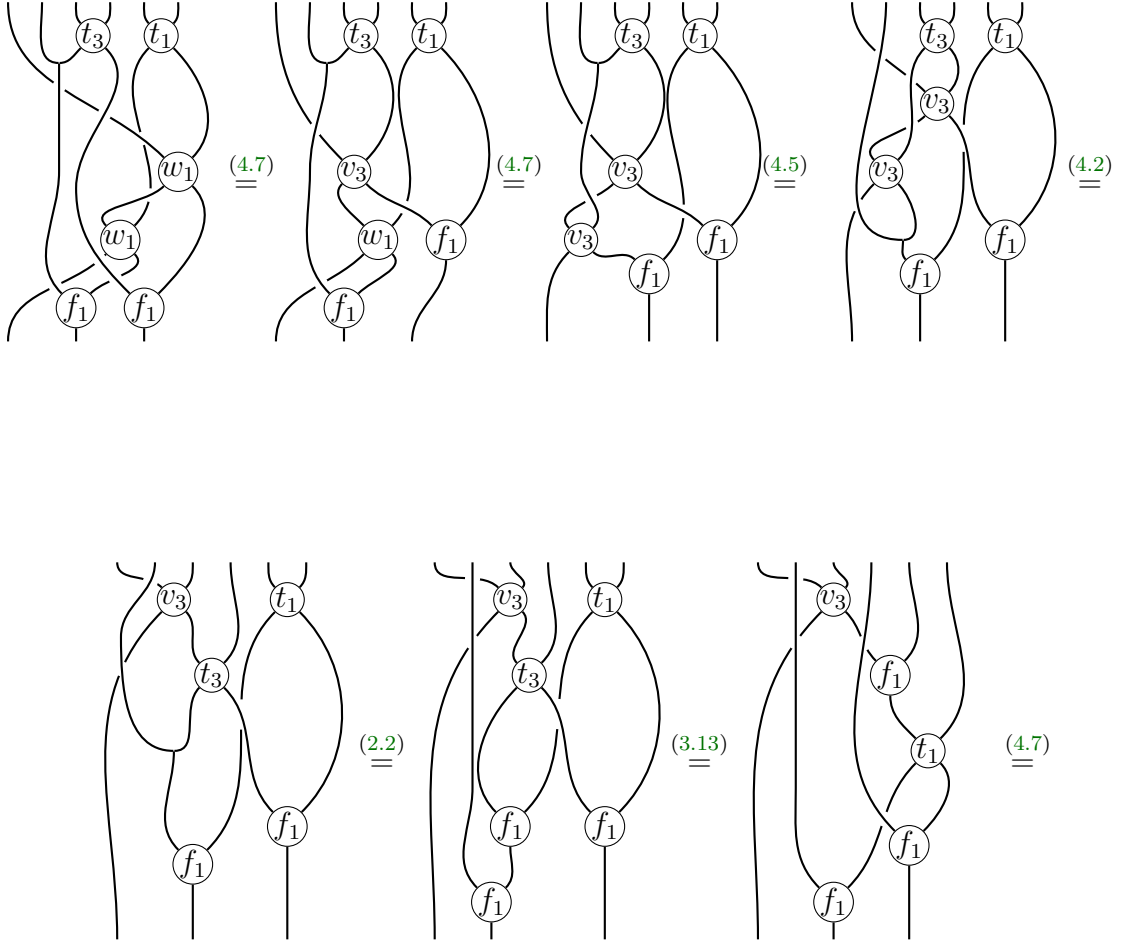
and the density of f .

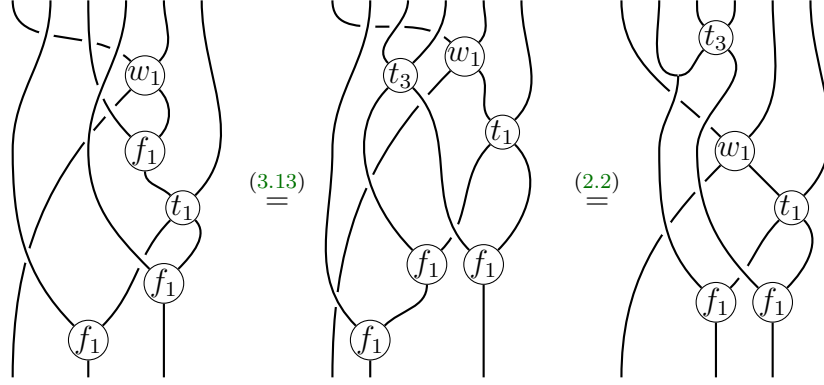
This proves (1); now we turn to (2). The counit condition follows easily by commutativity of the diagram

$$\begin{array}{ccccc}
 VAB & \xlongequal{\quad} & VAB & \xlongequal{\quad} & VAB & \xrightarrow{1f_1} & VB \\
 \downarrow 1f_1 & & \downarrow v^1 1 & & \downarrow 1e1 & & \downarrow 1e \\
 & & VAB & \xrightarrow{1e1} & VB & \xrightarrow{1e} & V \\
 & & \downarrow 1f_1 & & & & \parallel \\
 VB & \xrightarrow{w^1} & VB & \xrightarrow{1e} & V & &
 \end{array}$$

and density of f .

It remains only to prove the fusion law. For this, we use the fact that $m1.1t_3$ is in \mathcal{Q} , since by (3.11) it is equal to $d_2.1c^{-1}$. Now calculate as follows





and now cancel $m1.1t_3$ and use Remark 2.7.

The claim that a morphism of A -comodules is compatible also with the B -coaction induced by f , as well as the composition law of the resulting functors, are clear by the construction. The identity morphism induces the identity functor in light of (4.5). \square

Proposition 4.5 (1) can be applied in particular to the comodule (A, t_1, t_3) ; then any dense and multiplicative \mathbb{M} -morphism $f : A \rightharpoonup B$ induces morphisms as in the right verticals of

$$\begin{array}{ccc} A^2B & \xrightarrow{1f_1} & AB \\ t_1 1 \downarrow & & \downarrow t_1^f \\ A^2B & \xrightarrow{1f_1} & AB \end{array} \quad \begin{array}{ccc} A^2B & \xrightarrow{1c^{-1}} & ABA \xrightarrow{1f_2} AB \\ t_3 1 \downarrow & & \downarrow t_3^f \\ A^2B & \xrightarrow{1c^{-1}} & ABA \xrightarrow{1f_2} AB. \end{array}$$

If a pair (v^1, v^3) obeys (4.3) for the semigroup A then (v^3, v^1) obeys (4.3) for A^{op} in the braided monoidal category with inverse braiding. Thus by Proposition 4.5 (1), any dense and multiplicative \mathbb{M} -morphism $g : A^{\text{op}} \rightharpoonup B$ induces morphisms as in the right verticals of

$$\begin{array}{ccc} VAB & \xrightarrow{1g_1} & VB \\ v^3 1 \downarrow & & \downarrow (g_* v)^1 \\ VAB & \xrightarrow{1g_1} & VB \end{array} \quad \begin{array}{ccc} VAB & \xrightarrow{1c} & VBA \xrightarrow{1g_2} VB \\ v^1 1 \downarrow & & \downarrow (g_* v)^3 \\ VAB & \xrightarrow{1c} & VBA \xrightarrow{1g_2} VB. \end{array} \quad (4.8)$$

In particular, there are morphisms as in the right verticals of

$$\begin{array}{ccc} A^2B & \xrightarrow{1g_1} & AB \\ t_3 1 \downarrow & & \downarrow t_3^g \\ A^2B & \xrightarrow{1g_1} & AB \end{array} \quad \begin{array}{ccc} A^2B & \xrightarrow{1c} & ABA \xrightarrow{1g_2} AB \\ t_1 1 \downarrow & & \downarrow t_1^g \\ A^2B & \xrightarrow{1c} & ABA \xrightarrow{1g_2} AB. \end{array}$$

If A and B are bimonoids, then $(f_* v)^1$ in Proposition 4.5 and $(g_* v)^1$ above are mutually inverse isomorphisms in \mathcal{C} whenever f and g are mutual inverses for the so-called convolution product. For multiplier bimonoids, however, the convolution monoid is not available; instead, we have the following generalization.

Lemma 4.6. *Let $(A, t_1, t_2, t_3, t_4, e)$ and $(B, t_1, t_2, t_3, t_4, e)$ be regular multiplier bimonoids having the properties listed in Section 3.1. Let $f : A \rightharpoonup B$ and $g : A^{\text{op}} \rightharpoonup B$*

be dense multiplicative \mathbb{M} -morphisms. Using the construction and the notation of Proposition 4.5 and the previous paragraph, the following assertions are equivalent to each other.

- (a) $(g_*v)^1.(f_*v)^1 = 1$, for any comodule (v^1, v^3) .
- (b) $g_1.t_1^f = e1$.

Symmetrically, also the following assertions are equivalent to each other.

- (a') $(f_*v)^1.(g_*v)^1 = 1$, for any comodule (v^1, v^3) .
- (b') $f_1.t_3^g = e1$.

Proof. We only prove the equivalence of (a) and (b); the equivalence of the primed assertions follows by symmetry.

(a) \Rightarrow (b). By the density of g , the computation

implies that the bottom region of

$$\begin{array}{ccc}
 AB & \xrightarrow{t_1^f} & AB \\
 \parallel & \swarrow t_3^g & \downarrow g_1 \\
 AB & \xrightarrow{e1} & B
 \end{array}$$

commutes. The top region commutes by (a) applied to (t_1, t_3) .

(b) \Rightarrow (a). Using (b) in the first equality,

Using this in the second and the penultimate equalities, and writing w for f_*v and \tilde{w} for g_*v for brevity, we see that

where the fourth equality holds by the identity

which is obtained analogously to (3.20) from (4.2). Cancelling the epimorphisms $e111$ and $1f_1$ we obtain (a). \square

Corollary 4.7. *Let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier bimonoid in \mathcal{C} such that (A, t_1, t_2, e) is a multiplier Hopf monoid with non-degenerate multiplication and dense counit. Then the antipode $s: A^{\text{op}} \rightharpoonup A$ of Theorem 3.12 and the identity $i: A \rightharpoonup A$ are dense multiplicative \mathbb{M} -morphisms. It follows by (3.14) that $s_1.t_1 = e1$, so that by Lemma 4.6, $(s_*v)^1.v^1 = 1$ for any comodule (v^1, v^3) . On the other hand, we have the following series of equalities.*

Using the non-degeneracy of the multiplication and the density of s , we conclude that $m.t_3^s = e1$, so that by Lemma 4.6, also $v^1.(s_*v)^1 = 1$. That is to say, v^1 is an isomorphism in \mathcal{C} , for any comodule (v^1, v^3) .

4.8. Duals in the base category. Consider a braided monoidal category \mathcal{C} and an object V which possesses a left dual \bar{V} . We denote the unit and the counit of the duality by $\eta: I \rightarrow V\bar{V}$ and $\epsilon: \bar{V}V \rightarrow I$; since \mathcal{C} is braided, \bar{V} is also the left dual of V .

The duality induces a comonoid structure on the object $Q := \bar{V}V$ with comultiplication $\gamma: Q \rightarrow Q^2$ and counit $\zeta: Q \rightarrow I$ given by

$$\bar{V}V \xrightarrow{1\eta^1} \bar{V}V\bar{V}V \quad \bar{V}V \xrightarrow{\epsilon} I.$$

Using the duality, to give a morphism $v^1: VA \rightarrow VA$ is equivalently to give a morphism $q_1: QA = \bar{V}VA \rightarrow A$; explicitly, q_1 is given by the composite on the left below

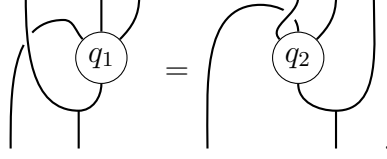
$$q_1: \bar{V}VA \xrightarrow{1v^1} \bar{V}VA \xrightarrow{\epsilon 1} A \quad v^1: VA \xrightarrow{\eta 11} V\bar{V}VA \xrightarrow{1q_1} VA \quad (4.9)$$

while one recovers v^1 from q_1 as the composite on the right. Similarly to give a morphism $v^3: VA \rightarrow VA$ is equivalently to give a morphism $q_2: AQ = A\bar{V}V \rightarrow A$, namely

$$A\bar{V}V \xrightarrow{c1} \bar{V}AV \xrightarrow{1c} \bar{V}VA \xrightarrow{1v^3} \bar{V}VA \xrightarrow{\epsilon 1} A.$$

Proposition 4.9. *Let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier bimonoid having the properties in Section 3.1. Morphisms $v^1, v^3: VA \rightarrow VA$ satisfy the compatibility condition (4.3) if and only if q_1 and q_2 satisfy the first condition of (2.1) and so are the components of an \mathbb{M} -morphism $q: Q \rightharpoonup A$. In this case v^1 and v^3 satisfy (4.1), and so make V into a comodule, if and only if q satisfies $e \bullet q = \zeta^\#$ and $qq \circ \gamma = d \bullet q$.*

Proof. Condition (4.3) takes the form



Applying the functor $\overline{V}(-)$ to it, and composing the result on the respective sides by

$$\overline{V}VA \xrightarrow{\epsilon^1} A \quad \text{and} \quad A\overline{V}VA \xrightarrow{c^{11}} \overline{V}AVA,$$

we obtain the equivalent form in (2.1).

The counit condition, appearing as the second half of (4.1), says that the diagram on the left

$$\begin{array}{ccc} QA & \xrightarrow{1e} & Q \\ q_1 \downarrow & & \downarrow \zeta \\ A & \xrightarrow{e} & I \end{array} \quad \begin{array}{ccc} AQ & \xrightarrow{\epsilon^1} & Q \\ q_2 \downarrow & & \downarrow \zeta \\ A & \xrightarrow{e} & I \end{array} \quad (4.10)$$

commutes. This says, in the language of \mathbb{M} -morphisms, that the first components of $e \bullet q$ and $\zeta^\# : A \rightarrowtail I$ are equal. Since the trivial semigroup (monoid, in fact) I is non-degenerate, this is equivalent to the equality of their second components — which appears on the right — and also to $e \bullet q = \zeta^\#$.

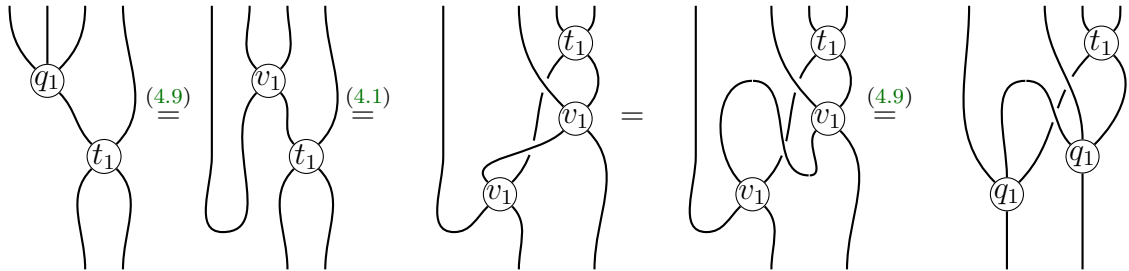
We have an \mathbb{M} -morphism $q: Q \rightarrowtail A$ and a dense multiplicative \mathbb{M} -morphism $d: A \rightarrowtail A^2$, and so can form the composite $d \bullet q: Q \rightarrowtail A^2$ which we shall call f ; it is defined by the commutativity of

$$\begin{array}{ccc} QA^3 & \xrightarrow{1d_1} & QA^2 \\ q_1^{11} \downarrow & & \downarrow f_1 \\ A^3 & \xrightarrow{d_1} & A^2. \end{array}$$

On the other hand we have the \mathcal{C} -morphism $\gamma: Q \rightarrow Q^2$ and the \mathbb{M} -morphism $qq: Q^2 \rightarrowtail A^2$, and they compose to give an \mathbb{M} -morphism $qq \circ \gamma: Q \rightarrowtail A^2$. This has first component

$$QA^2 \xrightarrow{\gamma^{11}} Q^2A^2 \xrightarrow{1c^1} (QA)^2 \xrightarrow{q_1q_1} A^2.$$

The first half of (4.1) asserts the equality of two morphisms $VA^2 \rightarrow VA^2$. Under the duality this can be expressed as the equality of two morphisms $\overline{V}VA^2 \rightarrow A^2$; specifically, the second and third morphisms appearing below.



Since the remaining equalities do hold, the first half of (4.1) is equivalent to the equality of the first and last expressions; writing in terms of Q rather than $\overline{V}V$ this says that the diagram on the left

$$\begin{array}{ccc} QA^2 & \xrightarrow{\gamma t_1} & Q^2 A^2 \xrightarrow{1c1} (QA)^2 \\ q_1 1 \downarrow & & \downarrow q_1 q_1 \\ A^2 & \xrightarrow{t_1} & A^2 \end{array} \quad \begin{array}{ccc} A^2 Q & \xrightarrow{t_2 \gamma} & A^2 Q^2 \xrightarrow{1c1} (AQ)^2 \\ 1q_2 \downarrow & & \downarrow q_2 q_2 \\ A^2 & \xrightarrow{t_2} & A^2 \end{array} \quad (4.11)$$

commutes; we shall also need its (equivalent by Lemma 4.2) counterpart for second components, which is the diagram on the right. The right A -linearity of v^1 , in the sense of (4.5), is equivalent to the right A -linearity of q_1 . By this and the non-degeneracy of A , commutativity of the first diagram in (4.11) amounts exactly to fact that f_1 is equal to $q_1 q_1 \cdot 1c1 \cdot \gamma 11$. Thus the fusion equation of (4.1) amounts to the identity $qq \circ \gamma = d \bullet q$. \square

If (V, v^1, v^3) and (W, w^1, w^3) are comodules whose underlying objects V and W have left duals \overline{V} and \overline{W} corresponding to \mathbb{M} -morphisms $q: \overline{V}V \rightarrow A$ and $q: \overline{W}W \rightarrow A$, then a morphism $f: V \rightarrow W$ is a morphism of comodules exactly when the diagram

$$\begin{array}{ccc} \overline{W}VA & \xrightarrow{1f1} & \overline{W}WA \\ \bar{f}11 \downarrow & & \downarrow q_1 \\ \overline{V}VA & \xrightarrow{q_1} & A \end{array} \quad (4.12)$$

commutes; here $\bar{f}: \overline{W} \rightarrow \overline{V}$ is the transpose of f .

For the same two comodules, we may form their monoidal product VW , as in [3]; in terms of \mathbb{M} -morphisms, the resulting comodule structure has first component

$$\overline{W}\overline{V}VWA \xrightarrow{c^{-1}111} \overline{V}\overline{W}VWA \xrightarrow{1c^{-1}11} \overline{V}V\overline{W}WA \xrightarrow{11q_1} \overline{V}VA \xrightarrow{q_1} A. \quad (4.13)$$

Theorem 4.10. *Consider a symmetric monoidal category \mathcal{C} which satisfies our standing assumptions of Section 2.1, and let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier bi-monoid in \mathcal{C} for which (A, t_1, t_2, e) is a multiplier Hopf monoid whose multiplication is non-degenerate and whose counit is dense. Let V be an object which possesses a left dual \overline{V} in \mathcal{C} . Then any comodule with underlying object V possesses a left dual in the monoidal category of comodules, with underlying object \overline{V} .*

Proof. By Remark 3.10, the comultiplication will also be dense, and so we may apply Proposition 4.9; thus in order to make \overline{V} into a comodule, it will suffice to define a suitable \mathbb{M} -morphism $q': Q' \rightarrow A$, where $Q' = V\overline{V}$, which is a comonoid in \mathcal{C} with comultiplication γ' and counit ζ' given by the following composites.

$$V\overline{V} \xrightarrow{1\eta 1} VV\overline{V}\overline{V} \xrightarrow{1c1} V\overline{V}V\overline{V} \quad V\overline{V} \xrightarrow{c^{-1}} \overline{V}V \xrightarrow{\epsilon} I$$

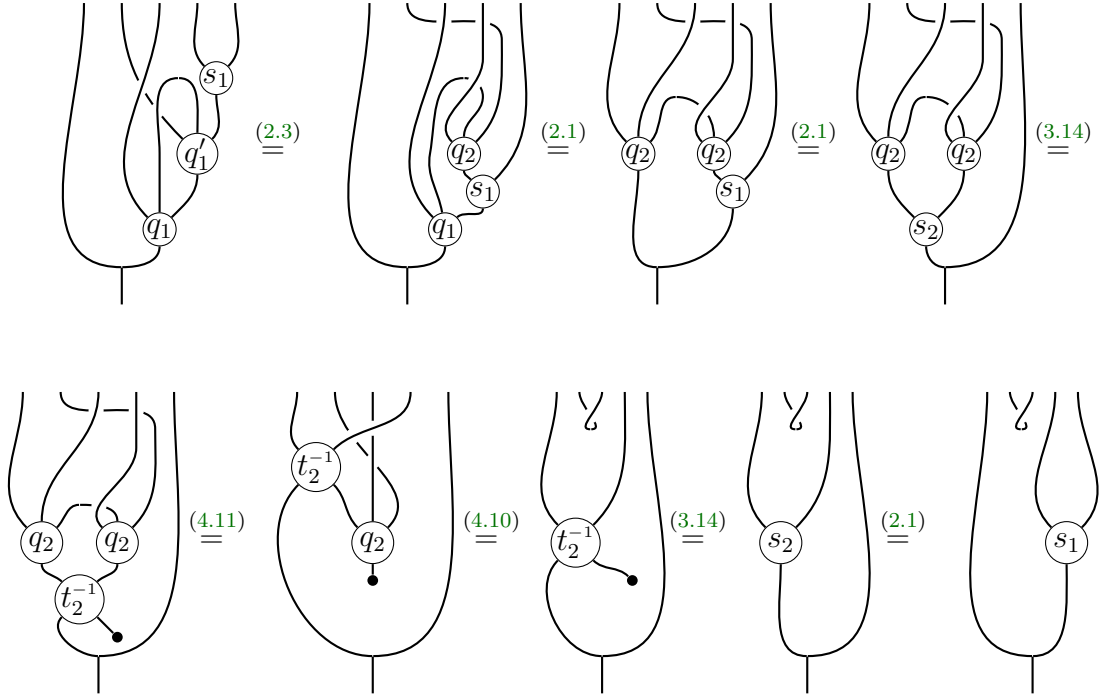
The inverse braiding $c^{-1}: V\overline{V} \rightarrow \overline{V}V$ defines a morphism $Q' \rightarrow Q$; in fact it is a comonoid morphism from the comonoid Q' to the opposite comonoid Q_{op} of Q . The desired \mathbb{M} -morphism q' is now given by $s \bullet q^{\text{op}} \circ c^{-1}$; we just need to check that this satisfies the conditions in Proposition 4.9. Compatibility with the comultiplication follows by the calculation

$$\begin{aligned}
d \bullet s \bullet q^{\text{op}} \circ c^{-1} &= ss \bullet c^{\#} \bullet d^{\text{op}} \bullet q^{\text{op}} \circ c^{-1} && \text{(Theorem 3.12)} \\
&= ss \bullet c^{\#} \bullet (d \bullet q)^{\text{op}} \circ c^{-1} && \text{(Proposition 2.11)} \\
&= ss \bullet c^{\#} \bullet (qq \circ \gamma)^{\text{op}} \circ c^{-1} && \text{(Proposition 4.9)} \\
&= ss \bullet c^{\#} \bullet (qq)^{\text{op}} \circ \gamma \circ c^{-1} && \text{(eq. (2.6))} \\
&= ss \bullet q^{\text{op}} q^{\text{op}} \circ c \circ \gamma \circ c^{-1} && \text{(eq. (2.6))} \\
&= ss \bullet q^{\text{op}} q^{\text{op}} \circ (c \cdot \gamma \cdot c^{-1}) && \text{(Section 2.6)} \\
&= ss \bullet q^{\text{op}} q^{\text{op}} \circ (c^{-1} c^{-1} \cdot \gamma') && \\
&= ss \bullet q^{\text{op}} q^{\text{op}} \circ c^{-1} c^{-1} \circ \gamma' && \text{(Section 2.6)} \\
&= (s \bullet q^{\text{op}} \circ c^{-1})(s \bullet q^{\text{op}} \circ c^{-1}) \circ \gamma' && \text{(Section 2.9)}
\end{aligned}$$

while compatibility with the counit is similar but easier:

$$\begin{aligned}
e \bullet s \bullet q^{\text{op}} \circ c^{-1} &= e^{\text{op}} \bullet q^{\text{op}} \circ c^{-1} && \text{(Theorem 3.12)} \\
&= (e \bullet q)^{\text{op}} \circ c^{-1} && \text{(Proposition 2.11)} \\
&= \zeta^{\# \text{op}} \circ c^{-1} && \text{(Proposition 4.9)} \\
&= \zeta^{\#} \circ c^{-1} && \\
&= (\zeta \cdot c^{-1})^{\#} && \text{(Section 2.6)} \\
&= \zeta'^{\#}.
\end{aligned}$$

It remains to show that the unit and counit of the duality are comodule morphisms. The first component of the structure morphism $(V\bar{V})^2 \rightarrow A$ of the monoidal product comodule $V\bar{V}$ is given by (4.13). Now



and now we may cancel s_1 and use non-degeneracy to deduce that the composite

$$V\bar{V}A \xrightarrow{c^{-1}1} \bar{V}VA \xrightarrow{11\eta^1} \bar{V}V^2\bar{V}A \xrightarrow{1c^{-1}11} \bar{V}V^2\bar{V}A \xrightarrow{11q'_1} \bar{V}VA \xrightarrow{q_1} A$$

is equal to $\epsilon 1.c^{-1}1: V\bar{V}A \rightarrow A$; but $\epsilon.c^{-1}$ is the transpose $\bar{\eta}$ of η , and so η satisfies the condition (4.12) to be a morphism of comodules.

Similarly the counit ϵ is a morphism of comodules by essentially the same calculation, with V and \bar{V} interchanged. \square

5. MODULES OVER MULTIPLIER HOPF MONOIDS

Modules over a regular multiplier bimonoid in a braided monoidal category were studied in [3]. In this section we look at what happens to this theory when the underlying multiplier bimonoid is Hopf.

5.1. Modules over non-degenerate semigroups. If (A, m) is a semigroup, we define a (right) A -module to be an object V equipped with an associative action $v: VA \rightarrow V$ which also lies in \mathcal{Q} . We shall sometimes refer to the property that $v \in \mathcal{Q}$ by saying that the action is *surjective*.

Remark 5.2. This surjectivity condition is supposed to capture the idea of unitality of an action in the absence of a unit for the multiplication; that is, in the context of a semigroup. If the multiplication $m: A^2 \rightarrow A$ possesses a unit $u: I \rightarrow A$, then the following properties of any associative action $v: VA \rightarrow V$ turn out to be equivalent:

- (a) v is unital; that is, $v.1u = 1$.
- (b) v is a split epimorphism.
- (c) v is a regular epimorphism.
- (d) v is an epimorphism.

Indeed, either (b) or (c) trivially implies (d), and (d) implies (a) by the commutativity of

$$\begin{array}{ccccc} & & VA & & \\ & \nearrow^{1u} & \uparrow^{v1} & \nwarrow^v & \\ & VA^2 & & & \\ & \nwarrow_{11u} & \searrow_{1m} & & \\ V & \xleftarrow{v} VA & \xrightarrow{11u} VA^2 & \xrightarrow{1m} VA & \xrightarrow{v} V \end{array}$$

The implication (a) \Rightarrow (b) is obvious and (a) implies that

$$\begin{array}{ccccc} VA^2 & \xrightarrow{v1} & VA & \xrightarrow{v} & V \\ & \searrow_{11u} & \swarrow_{1m} & \searrow_{1u} & \\ & & & & \end{array}$$

is a split coequalizer; hence (c).

We say that (V, v) is *non-degenerate* if for any objects X, Y of \mathcal{C} , the following map is injective.

$$\mathcal{C}(X, YV) \rightarrow \mathcal{C}(XA, YV), \quad f \mapsto XA \xrightarrow{f1} YVA \xrightarrow{1v} YV$$

Note that, here again as in Section 2.4, we made the strong assumption that this injectivity condition holds for *all* objects X and Y . As discussed in [1], for many

purposes it is enough, in fact, that it hold for all objects X , and all Y in a distinguished class as in Section 2.5.

Proposition 5.3. *Let $f: A \rightharpoonup B$ be a dense multiplicative \mathbb{M} -morphism between non-degenerate semigroups. There is an induced functor f^* sending non-degenerate B -modules to non-degenerate A -modules. For multiplicative and dense \mathbb{M} -morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, the composite functor g^*f^* is naturally isomorphic to $(g \bullet f)^*$ and the identity \mathbb{M} -morphism $A \rightharpoonup A$ induces the identity functor. If we regard \mathcal{M} as a bicategory with only identity 2-cells, then this defines a pseudofunctor from \mathcal{M} to the 2-category of categories, functors and natural transformations.*

Proof. Let $v: VB \rightarrow V$ be a surjective non-degenerate associative action of B . Then there is a coequalizer

$$X \xrightleftharpoons[x_2]{x_1} VB \xrightarrow{v} V$$

which is preserved by monoidal product. In the diagram

$$\begin{array}{ccccc}
 & & VB^2 & \xrightarrow{v1} & VB \\
 & \nearrow 1f_21 & & \searrow 1m & \downarrow v \\
 XAB & \xrightarrow{x_i11} & VBAB & & VB \xrightarrow{v} V \\
 & \searrow 1f_1 & \nearrow 11f_1 & \nearrow 1m & \uparrow v \\
 & & XB & \xrightarrow{x_i1} & VB^2 \xrightarrow{v1} VB
 \end{array}
 \quad (2.1)$$

the lower path depends only on $v.x_i$ not on i , thus the same is true of the upper path. By non-degeneracy of v , it follows that $v.1f_2.x_i1$ depends only on $v.x_i$, and so that there is a unique morphism f^*v making the first diagram of

$$\begin{array}{ccc}
 VBA & \xrightarrow{v1} & VA \\
 1f_2 \downarrow & & \downarrow f^*v \\
 VB & \xrightarrow{v} & V
 \end{array}
 \quad
 \begin{array}{ccc}
 VAB & \xrightarrow{1f_1} & VB \\
 (f^*v)1 \downarrow & & \downarrow v \\
 VB & \xrightarrow{v} & V
 \end{array}
 \quad (5.1)$$

commute. This f^*v also makes the the second diagram of (5.1) commute, by surjectivity of v and commutativity of the following diagrams.

$$\begin{array}{ccc}
 & VAB & \xrightarrow{(f^*v)1} VB \\
 v11 \nearrow & & \searrow v \\
 VBAB & \xrightarrow{1f_21} & VB^2 \\
 11f_1 \searrow & & \nearrow 1m \\
 & VB^2 & \xrightarrow{1m} VB
 \end{array}
 \quad
 \begin{array}{ccc}
 & VAB & \\
 v11 \nearrow & & \searrow 1f_1 \\
 VBAB & & VB \xrightarrow{v} V \\
 11f_1 \searrow & & \nearrow v \\
 & VB^2 & \xrightarrow{1m} VB
 \end{array}$$

Associativity and surjectivity of f^*v follow easily from the corresponding facts about v , but non-degeneracy requires a little more work. We must show that a morphism $g: X \rightarrow YV$ may be recovered from

$$XA \xrightarrow{g1} YVA \xrightarrow{1(f^*v)} YV.$$

But from this last composite we may construct the common composite of the diagram

$$\begin{array}{ccccc}
 XAB & \xrightarrow{g11} & YVAB & \xrightarrow{1(f^*v)1} & YVB \\
 1f_1 \downarrow & & 11f_1 \downarrow & (5.1) & \downarrow 1v \\
 XB & \xrightarrow{g1} & YVB & \xrightarrow{1v} & YV,
 \end{array}$$

and now by density of f we may recover $1v.g1$, and so finally by non-degeneracy of v we may recover g . The claim that a morphism of B -modules is also compatible with the A -action induced by f , as well as the composition law of the resulting functors, are clear by the construction. The identity morphism $B \rightarrow B$ induces the identity functor in light of the associativity of the actions $v : VB \rightarrow V$. \square

5.4. Modules over counital fusion morphisms. Let $t_2 : A^2 \rightarrow A^2$ and $e : A \rightarrow I$ satisfy equations (3.2). Typically, we shall be interested in the case where this structure is part of a multiplier bimonoid (A, t_1, t_2, e) .

An *action* of the fusion morphism $(A, t_2 : A^2 \rightarrow A^2)$ on an object V — a t_2 -action, for short — is a morphism $v_2 : VA \rightarrow VA$ which renders commutative the diagram

$$\begin{array}{ccccccc}
 VA^2 & \xrightarrow{v_2 1} & VA^2 & \xrightarrow{1c} & VA^2 & \xrightarrow{v_2 1} & VA^2 & \xrightarrow{1c^{-1}} & VA^2 \\
 1t_2 \downarrow & & & & & & & & \downarrow 1t_2 \\
 VA^2 & \xrightarrow{\quad \quad \quad v_2 1 \quad \quad \quad} & VA^2 & & & & & & VA^2.
 \end{array} \tag{5.2}$$

It follows that the morphism $v := 1e.v_2 : VA \rightarrow V$ is an associative action of the semigroup (A, m) ; if moreover it is an element of \mathcal{Q} , we say that (V, v_2) is a *module* over the counital fusion morphism (A, t_2, e) .

Remark 5.5. This generalizes the definition of module given in [3], where v was required to be a *split* epimorphism, while here we only ask that it lie in the class \mathcal{Q} . Of course this is no difference at all if \mathcal{Q} consists of the split epimorphisms. See also Remark 5.2 above.

Thus every module over the counital fusion morphism (A, t_2, e) has an underlying module over the semigroup $(A, 1e.t_2)$; in the non-degenerate case, these two notions turn out to be equivalent.

Proposition 5.6. *Let (A, t_2, e) be a counital fusion morphism with non-degenerate multiplication $m := 1e.t_2$. Then the forgetful functor from the category of (A, t_2, e) -modules to the category of (A, m) -modules is an isomorphism.*

Proof. For any (A, t_2, e) -module $v_2 : VA \rightarrow VA$, take the diagram (5.2) and compose with $1e1 : VA^2 \rightarrow VA$, and use the second half of (3.2) to deduce the commutativity of

$$\begin{array}{ccc}
 VA^2 & \xrightarrow{v1} & VA \\
 1t_2 \downarrow & & \downarrow v_2 \\
 VA^2 & \xrightarrow{v1} & VA.
 \end{array} \tag{5.3}$$

Since $v1$ is a (regular) epimorphism we see that there can be at most one t_2 -action v_2 for a given surjective semigroup action v . For the converse, we have a coequalizer

$$X \xrightarrow[x_2]{x_1} VA \xrightarrow{v} V$$

which is preserved by the monoidal product. In the commutative diagram

$$\begin{array}{ccccc}
 & & VA^3 & \xrightarrow{v11} & VA^2 \\
 & \nearrow^{1t_21} & & \searrow^{11m} & \nearrow^{1m} \\
 XA^2 & \xrightarrow{x_i11} & VA^3 & & VA^2 \xrightarrow{v1} VA \\
 & \searrow^{1t_1} & & \nearrow^{1m1} & \nearrow^{v1} \\
 & & XA^2 \xrightarrow{x_i11} & VA^3 \xrightarrow{v11} & VA^2
 \end{array} \quad (3.7)$$

the lower composite depends only on $v.x_i$, not on i , thus the same is true of the upper composite. It follows by non-degeneracy of m that $v1.1t_2.x_i1$ depends only on $v.x_i$ and so finally that there is a (unique) v_2 making (5.3) commute. We re-obtain v from this v_2 as $1e.v_2$ since the morphisms of its top row are epimorphisms, and the following diagram commutes.

$$\begin{array}{ccc}
 VA & \xleftarrow{v1} & VA^2 \xrightarrow{v1} VA \\
 & \nearrow^{1t_2} & \searrow^{1m} \\
 v_2 \downarrow & VA^2 \xrightarrow{11e} & VA \downarrow v \\
 & \downarrow^{v1} & \downarrow v \\
 VA & \xrightarrow{1e} & V
 \end{array}$$

The fact that this v_2 makes (5.2) commute follows immediately from the fusion equation (3.2) and the fact that $v11: VA^3 \rightarrow VA^2$ is an epimorphism.

This now proves that we have a bijection on objects. It is clear that any morphism of (A, t_2, e) -modules respects the induced (A, m) -actions; the converse follows by (5.3) and surjectivity of the (A, m) -action. \square

If (V, v) and (W, w) are modules over a non-degenerate semigroup $(A, 1e.t_2)$ associated to a counital fusion morphism (A, t_2, e) , there is an $(A, 1e.t_2)$ -action ψ on the monoidal product VW given by the composite

$$VWA \xrightarrow{1w_2} VWA \xrightarrow{1c^{-1}} VAW \xrightarrow{v1} VW \quad (5.4)$$

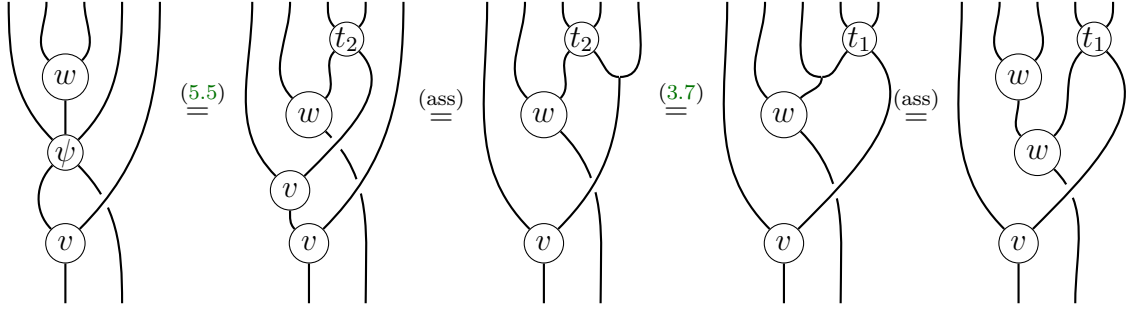
where $w_2: WA \rightarrow WA$ is defined using w as in Proposition 5.6. By the defining property (5.3) of w_2 , it follows that ψ makes the diagram

$$\begin{array}{ccc}
 VW A^2 & \xrightarrow{11t_2} & VW A^2 \xrightarrow{1w1} VWA \xrightarrow{1c^{-1}} VAW \\
 \downarrow^{1w1} & & \downarrow^{v1} \\
 VW A & \xrightarrow{\psi} & VW
 \end{array} \quad (5.5)$$

commute. In general there is no reason why ψ should lie in \mathcal{Q} , but it clearly will do so if t_2 is in \mathcal{Q} . In fact, if the multiplication $1e.t_2$ belongs to \mathcal{Q} — so that it renders A

itself a module — then ψ lies in \mathcal{Q} for any modules V and W if and only if the induced morphism d_2 of (3.8) does so: see [3, Corollary 2.23]. Similarly, the counit e induces an $(A, 1e.t_2)$ -action on the monoidal unit I , and this will be a module provided that $e \in \mathcal{Q}$. When both d_2 and e lie in \mathcal{Q} , then the category of modules will be monoidal, as in [3, Corollary 2.23] once again.

5.7. Modules over multiplier bimonoids. A module over a multiplier bimonoid (A, t_1, t_2, e) will just mean a module over the underlying counital fusion morphism (A, t_2, e) . In this case, however, there is an alternative characterization of the action $\psi: VWA \rightarrow VW$ of (5.4): by the calculation



and surjectivity of w , we may deduce that the diagram

$$\begin{array}{ccccc} VW A^2 & \xrightarrow{1t_1} & VW A^2 & \xrightarrow{1w1} & VW A & \xrightarrow{1c^{-1}} & VAW \\ \psi 1 \downarrow & & & & & & \downarrow v1 \\ VW A & \xrightarrow{1c^{-1}} & VAW & \xrightarrow{v1} & VW & & \end{array} \quad (5.6)$$

commutes, and by non-degeneracy of v it follows that this characterizes ψ .

5.8. Modules over regular multiplier bimonoids. Now consider a regular multiplier bimonoid $(A, t_1, t_2, t_3, t_4, e)$. A module [3] over this multiplier bimonoid is a tuple (V, v_2, v_3) where $v_2: VA \rightarrow VA$ makes V into a module over (A, t_2, e) and $v_3: AV \rightarrow AV$ renders commutative the analogue

$$\begin{array}{ccccccc} A^2V & \xrightarrow{1v_3} & A^2V & \xrightarrow{c^{-1}1} & A^2V & \xrightarrow{1v_3} & A^2V & \xrightarrow{c1} & A^2V \\ t_3 1 \downarrow & & & & & & & & \downarrow t_3 1 \\ A^2V & \xrightarrow{1v_3} & A^2V & & & & & & A^2V \end{array} \quad (5.7)$$

of (5.2), subject to the compatibility conditions expressed by the commutativity of

$$\begin{array}{ccc} VA^2 & \xrightarrow{v_2 1} & VA^2 \\ t_3 \downarrow & & \downarrow t_3 \\ VA^2 & \xrightarrow{v_2 1} & VA^2 \end{array} \quad \begin{array}{ccc} A^2V & \xrightarrow{1v_3} & A^2V \\ t_2 1 \downarrow & & \downarrow t_2 1 \\ A^2V & \xrightarrow{1v_3} & A^2V \end{array} \quad \begin{array}{ccc} VA & \xrightarrow{v_2} & VA \\ c \downarrow & & \downarrow 1e \\ AV & \xrightarrow{v_3} & AV \\ v_3 \downarrow & & \downarrow e1 \\ AV & \xrightarrow{e1} & V. \end{array} \quad (5.8)$$

(There is no need to impose separately a surjectivity condition on $e1.v_3$, since this follows from surjectivity for $1e.v_2$ and the last compatibility condition.)

A morphism of modules is just a morphism in the underlying category, satisfying the evident compatibility conditions with v_2 and v_3 .

We now have the following extension of Proposition 5.6.

Proposition 5.9. *The category of modules of a regular multiplier bimonoid $(A, t_1, t_2, t_3, t_4, e)$ with non-degenerate multiplication $m := e1.t_1$ is isomorphic to the category of modules of the underlying semigroup (A, m) .*

Proof. We have already seen that there is a unique t_2 -action v_2 corresponding to any surjective associative action $v : VA \rightarrow V$. Dually, there is a unique v_3 making the diagram

$$\begin{array}{ccccc} A^2V & \xrightarrow{1c^{-1}} & AVA & \xrightarrow{1v} & AV \\ t_3 1 \downarrow & & & & \downarrow v_3 \\ A^2V & \xrightarrow{1c^{-1}} & AVA & \xrightarrow{1v} & AV \end{array} \quad (5.9)$$

commute, and furthermore this now implies that (5.7) commutes.

As for (5.8), the first two diagrams commute by commutativity of the last diagram in (3.10) and surjectivity of the semigroup action v , while the third commutes since both paths yield v by the construction of v_2 and v_3 .

This now proves that we have a bijection on objects. It is clear that any morphism of modules over the regular multiplier bimonoid respects the induced semigroup actions; the converse follows by (5.3), (5.9), and surjectivity of the semigroup action. \square

5.10. Duals in the base category.

Proposition 5.11. *Let A be a semigroup in a monoidal category. Let V be an object with a left dual \overline{V} , and let $w : AV \rightarrow V$ be a surjective associative (left) action. Then the action $v : \overline{V}A \rightarrow \overline{V}$ given by*

$$\overline{V}A \xrightarrow{11\eta} \overline{V}AV\overline{V} \xrightarrow{1w1} \overline{V}V\overline{V} \xrightarrow{\epsilon 1} \overline{V}$$

is associative and non-degenerate. If moreover w is non-degenerate, then v is an epimorphism, preserved by the monoidal product.

Proof. Associativity follows easily from associativity of w ; this is probably easiest done using string diagrams. As for non-degeneracy, we must show that, for any objects X and Y , the morphism $g : X \rightarrow Y\overline{V}$ can be recovered from

$$XA \xrightarrow{g1} Y\overline{V}A \xrightarrow{1v} Y\overline{V}.$$

From $1v.g1$ we may form the common composite of

$$\begin{array}{ccccc} XAV & \xrightarrow{g11} & Y\overline{V}AV & \xrightarrow{1v1} & Y\overline{V}V \\ 1w \downarrow & & 11w \downarrow & & \downarrow 1\epsilon \\ XV & \xrightarrow{g1} & Y\overline{V}V & \xrightarrow{1\epsilon} & Y \end{array}$$

and since $1w$ is epimorphic this allows us to recover the composite $1\epsilon.g1$ along the bottom, and this in turn allows us to recover g using the duality.

Now suppose that w is non-degenerate, and let $x_1, x_2: \bar{V}Y \rightarrow Z$ satisfy $x_1.v1 = x_2.v1$. By commutativity of the diagram

$$\begin{array}{ccccccc}
 \bar{V}AY & \xrightarrow{11\eta 1} & \bar{V}AV\bar{V}Y & \xrightarrow{111x_i} & \bar{V}AVZ & \xrightarrow{1w1} & \bar{V}VZ \\
 \downarrow v1 & & & & \downarrow v11 & & \downarrow \epsilon 1 \\
 \bar{V}Y & \xrightarrow{1\eta 1} & \bar{V}V\bar{V}Y & \xrightarrow{11x_i} & \bar{V}VZ & & \\
 \parallel & \swarrow \epsilon 11 & & & \searrow \epsilon 1 & & \\
 \bar{V}Y & & & \xrightarrow{x_i} & & & Z
 \end{array}$$

we see that the upper composite is independent of i ; using the duality once it follows that $w1.11x_i.1\eta 1$ is independent of i , then by non-degeneracy of w it follows that $1x_i.\eta 1$ is independent of i , and finally using duality again it follows that x_i is independent of i ; in other words, that $x_1 = x_2$. \square

Take now a multiplier bimonoid (A, t_1, t_2, e) with non-degenerate multiplication and dense counit. Assume that it is Hopf, with antipode $s: A^{\text{op}} \rightharpoonup A$. Let V be an object possessing a left dual \bar{V} , and a surjective associative non-degenerate action $v: VA \rightarrow V$. Then the action $s^*v: VA^{\text{op}} \rightarrow V$ of Proposition 5.3 may be regarded as a left action $AV \xrightarrow{c^{-1}} VA \xrightarrow{s^*v} V$, and so by Proposition 5.11 we obtain an action $\bar{v}: \bar{V}A \rightarrow \bar{V}$; explicitly, this is given by

$$\bar{V}A \xrightarrow{11\eta} \bar{V}AV\bar{V} \xrightarrow{1c^{-1}1} \bar{V}VA\bar{V} \xrightarrow{1(s^*v)1} \bar{V}V\bar{V} \xrightarrow{\epsilon 1} \bar{V}. \quad (5.10)$$

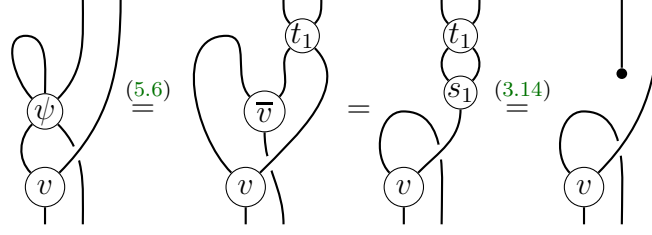
Theorem 5.12. *Let \mathcal{C} be a symmetric monoidal category satisfying our standing assumptions of Section 2.1. Let (A, t_1, t_2, e) be a multiplier Hopf monoid whose multiplication $m := e1.t_1$ is non-degenerate and whose counit is dense. Finally, let V be a module over the multiplier bimonoid (A, t_1, t_2, e) whose underlying (A, m) -action $v: VA \rightarrow V$ is non-degenerate. If V has a left dual \bar{V} in \mathcal{C} and the morphism \bar{v} in (5.10) is in \mathcal{Q} , then it makes \bar{V} into an (A, t_1, t_2, e) -module which is left dual to V .*

Proof. By Proposition 5.11 and the assumption that \bar{v} is in \mathcal{Q} , \bar{v} makes \bar{V} a module over the semigroup (A, m) and so by Proposition 5.6 a module over the multiplier bimonoid (A, t_1, t_2, e) . We need only show that the unit and counit of the duality are morphisms of (A, m) -modules.

In the case of the unit, the key is the following calculation linking v and \bar{v} .

If ψ is the action for the monoidal product $V\bar{V}$, then, using this last calculation and the fact that the braiding is a symmetry for the second step of the next calculation,

we have

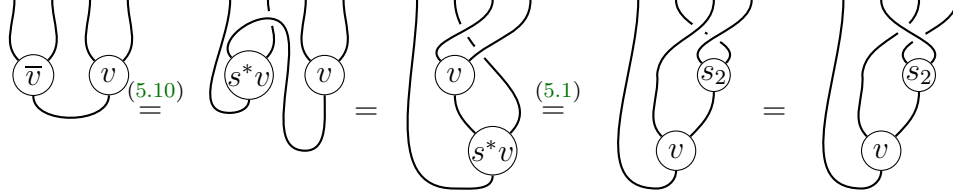


and so by, non-degeneracy of v , the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta^1} & V\overline{V}A \\ e \downarrow & & \downarrow \psi \\ I & \xrightarrow{\eta} & V\overline{V} \end{array}$$

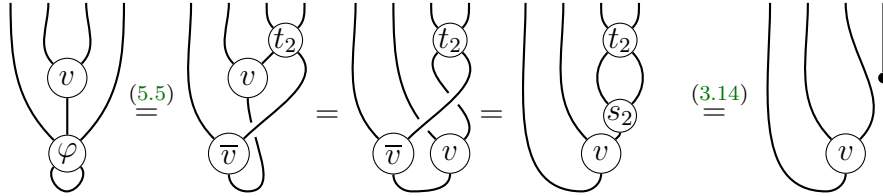
commutes, and so η is a morphism of modules.

In the case of the counit, once again we start with a calculation linking v and \overline{v} .



where the last step uses the fact that the braiding is a symmetry.

If φ is the action for the monoidal product $\overline{V}V$, then, using this last calculation for the third step, we have



and so by the surjectivity of v the diagram

$$\begin{array}{ccc} \overline{V}VA & \xrightarrow{\epsilon^1} & A \\ \varphi \downarrow & & \downarrow e \\ \overline{V}V & \xrightarrow{\epsilon} & I \end{array}$$

commutes, and so ϵ is a morphism of modules. \square

Remark 5.13. In Theorem 5.12 we have had to assume that the action \overline{v} lies in \mathcal{Q} ; otherwise, the hypotheses only guarantee that it is an epimorphism. If the base category is abelian, and so in particular if it is the category of modules over a (commutative) ring, then every epimorphism is a regular epimorphism; thus if moreover \mathcal{Q} consists of all regular epimorphisms, then the action \overline{v} will always be surjective.

6. THE FUNDAMENTAL THEOREM OF HOPF MODULES

For a regular multiplier bimonoid A in a braided monoidal category \mathcal{C} having the properties in Section 3.1, a Hopf module is defined as an object V equipped with A -module and A -comodule structures, subject to a compatibility condition requiring that the action $VA \rightarrow V$ is a morphism of comodules. Taking the monoidal product with A induces a functor from \mathcal{C} to the category of Hopf modules. When it is an equivalence, A is said to satisfy the *fundamental theorem of Hopf modules*. In this section we investigate when this happens.

We shall say that a monomorphism is A -pure if it is preserved by taking the monoidal product with A .

Definition 6.1. For a regular multiplier bimonoid $(A, t_1, t_2, t_3, t_4, e)$ in a braided monoidal category \mathcal{C} having the properties in Section 3.1, a *Hopf module* is a tuple (V, v^1, v^3, v_2, v_3) where (V, v^1, v^3) is a comodule as in Section 4.1, (V, v_2, v_3) is a module as in Section 5.8, and either (hence by Lemma 4.3 both) of the so-called Hopf compatibility conditions – saying that $v := 1e.v_2: VA \rightarrow V$ is a morphism of comodules – hold:

$$\begin{array}{ccc}
 VA^2 & \xrightarrow{1t_1} VA^2 & \xrightarrow{c1} AVA & \xrightarrow{1v^1} AVA & \xrightarrow{c^{-1}1} VA^2 \\
 v^1 \downarrow & & & & \downarrow v^1 \\
 VA & \xrightarrow{\quad \quad \quad v^1 \quad \quad \quad} & VA & &
 \end{array} \tag{6.1}$$

$$\begin{array}{ccccccc}
 VA^2 & \xrightarrow{1c^{-1}} VA^2 & \xrightarrow{v^31} VA^2 & \xrightarrow{1c} VA^2 & \xrightarrow{1t_3} VA^2 & & \\
 v^1 \downarrow & & & & \downarrow v^1 & & \\
 VA & \xrightarrow{\quad \quad \quad v^3 \quad \quad \quad} & VA & & VA & &
 \end{array}$$

A morphism of Hopf modules $(V, v^1, v^3, v_2, v_3) \rightarrow (W, w^1, w^3, w_2, w_3)$ is a morphism $V \rightarrow W$ in \mathcal{C} which is both a morphism of modules and a morphism of comodules.

Proposition 6.2. Let \mathcal{C} be a braided monoidal category satisfying our standing assumptions of Section 2.1, and $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier bimonoid in \mathcal{C} having the properties in Section 3.1. For any object X of \mathcal{C} , there is a Hopf module XA with comodule structure

$$(XA^2 \xrightarrow{1t_1} XA^2, XA^2 \xrightarrow{1t_3} XA^2)$$

and module structure

$$(XA^2 \xrightarrow{1t_2} XA^2, AXA \xrightarrow{c^{-1}1} XA^2 \xrightarrow{1t_3} XA^2 \xrightarrow{c1} AXA).$$

This is the object part of a functor $(-)A$ from \mathcal{C} to the category of A -Hopf modules.

Proof. By the fusion and counit equations for t_1 , the morphism $1t_1: XA^2 \rightarrow XA^2$ equips XA with the structure of a comodule over t_1 , in the sense that the diagrams of (4.1) commute. Similarly, $1t_3$ makes XA into a comodule over t_3 . The compatibility condition (4.3) holds by the second diagram of (3.10).

The morphism $1t_2: XA^2 \rightarrow XA^2$ renders commutative the diagram of (5.2) by the fusion equation on t_2 ; and $c1.1t_3.c^{-1}1: AXA \rightarrow AXA$ renders commutative (5.7) by

the fusion equation on t_3 . The first and the second diagrams of (5.8) commute by the last diagram of (3.10). By the first diagram of (3.10), both paths around the last diagram of (5.8) are equal to $1m : XA^2 \rightarrow XA$ in \mathcal{Q} .

The Hopf module compatibility conditions in (6.1) hold by the short fusion equation (3.5) on t_1 and t_3 , respectively. \square

Recall that the monoidal category \mathcal{C} is said to be *left closed* if for any object X , the endofunctor $X(-)$ possesses a right adjoint, to be denoted by $[X, -]$. Whenever \mathcal{C} is braided, then $[X, -]$ is also right adjoint to the functor $(-)X$, meaning that \mathcal{C} is also *right closed*; so without any confusion we can just call it *closed*. The unit and the counit of the adjunction $X(-) \dashv [X, -]$ will be denoted by η and ϵ , respectively.

Recall from Corollary 4.7 that for a comodule (V, v^1, v^3) over a multiplier Hopf monoid with non-degenerate multiplication and dense counit, the morphism v^1 is invertible. In particular, we may apply this to (the underlying comodule of) a Hopf module.

Lemma 6.3. *Let \mathcal{C} be a closed braided monoidal category satisfying our standing assumptions of Section 2.1. Let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier bimonoid in \mathcal{C} with non-degenerate multiplication, dense counit, and such that (A, t_1, t_2, e) is a multiplier Hopf monoid. Assume that for any Hopf module (V, v^1, v^3, v_2, v_3) , the morphism*

$$w := V \xrightarrow{\eta} [A, AV] \xrightarrow{[A, c]} [A, VA] \xrightarrow{[A, (v^1)^{-1}]} [A, VA] \xrightarrow{[A, v]} [A, V]$$

factorizes as $V \xrightarrow{p} V^c \xrightarrow{i} [A, V]$, where p is an epimorphism and i is an A -pure monomorphism. Then the following hold.

(1) *There is a unique morphism $\tilde{n} : V \rightarrow V^c A$ rendering commutative*

$$\begin{array}{ccc} VA & \xrightarrow{v} & V \\ v^1 \downarrow & & \downarrow \tilde{n} \\ VA & \xrightarrow{p1} & V^c A \end{array} .$$

(2) *The morphism \tilde{n} in part (1) is the inverse of*

$$n := V^c A \xrightarrow{i1} [A, V]A \xrightarrow{c^{-1}} A[A, V] \xrightarrow{\epsilon} V . \quad (6.2)$$

Proof. (1) By assumption, $v : VA \rightarrow V$ arises as the coequalizer of some morphisms f and $g : X \rightarrow VA$; thus it is enough to prove that $p1.v^1$ coequalizes f and g . Since $i1 : V^c A \rightarrow [A, V]A$ is a monomorphism by assumption, this is equivalent to showing that $w1.v^1$ coequalizes f and g . This requires some preparation.

In the following stringy computation the undecorated bubble stands for $(v^1)^{-1}$.

This proves that

$$\begin{array}{ccccc}
 VA^2 & \xrightarrow{1e1} & VA & \xrightarrow{(v^1)^{-1}} & VA \\
 v1 \downarrow & & & & \downarrow v \\
 VA & \xrightarrow{(v^1)^{-1}} & VA & \xrightarrow{v} & V
 \end{array} \quad (6.3)$$

commutes, and now the commutativity of the next diagram also follows.

$$\begin{array}{ccccccc}
 VA^2 & = & VA^2 & = & VA^2 & \xrightarrow{v^1 1} & VA^2 \xrightarrow{\eta 11} [A, AV]A^2 \xrightarrow{[A, c]11} [A, VA]A^2 \\
 \downarrow & & \downarrow 1t_1 & & \downarrow 1m & & \downarrow 1m \\
 & & VA^2 & & & & [A, (v^1)^{-1}]11 \\
 & & \downarrow c1 & \searrow 1e1 & & & \downarrow \\
 & & AVA & \xrightarrow{e11} & VA & \xrightarrow{v^1} & VA \\
 & & \downarrow 1v^1 & & & & \parallel \\
 & & AVA & \xrightarrow{e11} & VA & & \parallel \\
 & & \downarrow c^{-1}1 & & & & \parallel \\
 & & VA^2 & \xrightarrow{1e1} & VA & & \parallel \\
 & & \parallel & & \downarrow \eta 1 & & \parallel \\
 & & VA^2 & \xrightarrow{\eta 1} [A, AVA]A = [A, AVA]A \xrightarrow{[A, 11e]1} [A, AV]A & & & \parallel \\
 & & \downarrow v1 & & \downarrow [A, c_A, VA]1 & \downarrow [A, c]1 & \parallel \\
 (6.1) & & VA & \xrightarrow{v^1} VA \xrightarrow{\eta 1} [A, AV]A \xrightarrow{[A, c]1} [A, VA]A \xrightarrow{[A, (v^1)^{-1}]1} [A, AV]A \xrightarrow{[A, v]1} [A, V]A & & & \parallel \\
 & & & & \downarrow [A, 1v]1 & \downarrow [A, 1e1]1 & \downarrow [A, (v^1)^{-1}]1 & \downarrow [A, v]1 \\
 & & & & [A, VA^2]A \xrightarrow{[A, v1]1} [A, VA]A \xrightarrow{[A, (v^1)^{-1}]1} [A, VA]A \xrightarrow{[A, v]1} [A, V]A & & & \parallel \\
 & & & & & & (6.3) & \parallel
 \end{array}$$

Since the left-bottom path coequalizes $f1$ and $g1$, so does the top-right path; from which we conclude using the non-degeneracy of $m = e1.t_1$.

(2) Since $v : VA \rightarrow V$ is an epimorphism by assumption, it follows by the commutativity of

$$\begin{array}{c}
 \begin{array}{ccccccc}
 VA & \xrightarrow{\quad v \quad} & & & & & V \\
 \downarrow v^1 & & & & & & \downarrow \tilde{n} \\
 VA & \xrightarrow{\quad p1 \quad} & & & & & V^c A \\
 \parallel & & & & & & \downarrow i1 \\
 VA & \xrightarrow{\eta^1} [A, AV]A \xrightarrow{[A, c]1} [A, VA]A \xrightarrow{[A, (v^1)^{-1}]1} [A, VA]A \xrightarrow{[A, v]1} [A, V]A \\
 \downarrow c^{-1} & & & & & & \downarrow c^{-1} \\
 AV & \xrightarrow{1\eta} A[A, AV] \xrightarrow{1[A, c]} A[A, VA] \xrightarrow{1[A, (v^1)^{-1}]} A[A, VA] \xrightarrow{1[A, v]} A[A, V] \\
 & \downarrow \epsilon & & & & & \downarrow \epsilon \\
 & AV & \xrightarrow{c} & VA & \xrightarrow{(v^1)^{-1}} & VA & \xrightarrow{v} & V
 \end{array} \\
 \text{with a curved arrow } n \text{ from } V^c A \text{ to } V \text{ and a curved arrow } \epsilon \text{ from } AV \text{ to } VA.
 \end{array}$$

that $n.\tilde{n} = 1$.

In order to compute $\tilde{n}.n$, consider the commutative diagram

$$\begin{array}{c}
 \begin{array}{ccccccc}
 VA^2 & \xrightarrow{\quad v^3 1 \quad} & & & & & VA^2 \\
 \downarrow 1s_1 & & & & & & \downarrow 1s_1 \\
 VA & \xrightarrow{\quad c^{-1} \quad} & AV & & & & VA \\
 \parallel & & \downarrow \eta^1 & & \downarrow 1\eta & & \parallel \\
 VA & & [A, AV]A & & A[A, AV] & & VA \\
 & & \downarrow [A, c]1 & & \downarrow 1[A, c] & & \\
 & & [A, VA]A & & A[A, VA] & & \\
 & & \downarrow [A, (v^1)^{-1}]1 & & \downarrow 1[A, (v^1)^{-1}] & & \\
 & & [A, VA]A & & A[A, VA] & & \\
 & & \downarrow [A, v]1 & & \downarrow 1[A, v] & & \\
 & & [A, V]A & & A[A, V] & & \\
 & \xrightarrow{i1} & \xrightarrow{c^{-1}} & & \xrightarrow{\epsilon} & & \\
 V^c A & & & & & & V
 \end{array} \\
 \text{with curved arrows } n \text{ from } V^c A \text{ to } V \text{ and } \epsilon \text{ from } AV \text{ to } VA.
 \end{array} \tag{6.4}$$

whose top region is seen to commute by applying (4.8) to $g = s$. With its help we see that

$$\begin{array}{ccccc}
 VA^2 & \xrightarrow{1s_1} & VA & & \\
 \downarrow 1s_1 & \searrow v^3 1 & \downarrow v^1 & & \downarrow p1 \\
 & VA^2 & \xrightarrow{1s_1} & VA & \\
 & \downarrow v & & & \\
 VA & \xrightarrow{p1} & V^c A & \xrightarrow{n} & V & \xrightarrow{\tilde{n}} & V^c A
 \end{array}$$

(6.4)

commutes; where commutativity of the region at the top follows postcomposing by v^1 the defining identity $(v^1)^{-1}.1s_1 = 1s_1.v^31$ of $(v^1)^{-1} = (s_*v)^1$ in (4.8). Since $1s_1$ and $p1$ are epimorphisms, this proves $\tilde{n}.n = 1$. \square

Theorem 6.4 (The fundamental theorem of Hopf modules). *Let \mathcal{C} be a closed braided monoidal category satisfying our standing assumptions of Section 2.1. Let $(A, t_1, t_2, t_3, t_4, e)$ be a regular multiplier bimonoid in \mathcal{C} with non-degenerate multiplication, dense counit, and such that (A, t_1, t_2, e) is a multiplier Hopf monoid. Assume that, for any Hopf module (V, v^1, v^3, v_2, v_3) , the morphism*

$$V \xrightarrow{\eta} [A, AV] \xrightarrow{[A, c]} [A, VA] \xrightarrow{[A, (v^1)^{-1}]} [A, VA] \xrightarrow{[A, v]} [A, V] \quad (6.5)$$

factorizes as $V \xrightarrow{p} V^c \xrightarrow{i} [A, V]$ through some object V^c , where p is a strong epimorphism and i is an A -pure monomorphism. Then the functor $(-)^c$ in Proposition 6.2, from \mathcal{C} to the category of Hopf modules, is an equivalence with inverse $(-)^c$.

Proof. Since the strong epi-mono factorization is unique up to isomorphism, the object part of the functor $(-)^c$ is well-defined. Concerning its action on the morphisms, we use the orthogonality property of the strong epimorphism p . For any morphism of Hopf modules $h : V \rightarrow W$, in the diagram

$$\begin{array}{ccccccc} & & & & V^c & & \\ & & & & \downarrow i & & \\ V & \xrightarrow{\eta} & [A, AV] & \xrightarrow{[A, c]} & [A, VA] & \xrightarrow{[A, (v^1)^{-1}]} & [A, VA] \xrightarrow{[A, v]} [A, V] \\ \downarrow h & & \downarrow [A, h1] & & \downarrow [A, h1] & & \downarrow [A, h] \\ W & \xrightarrow{\eta} & [A, AW] & \xrightarrow{[A, c]} & [A, WA] & \xrightarrow{[A, (w^1)^{-1}]} & [A, WA] \xrightarrow{[A, w]} [A, W] \\ & & & & W^c & & \\ & & & & \downarrow i & & \end{array}$$

p (curved arrow from V to V^c), p (curved arrow from W to W^c)

the leftmost region commutes by naturality. The middle region commutes since h is a morphism of comodules and the region on the right commutes because h is a morphism of modules. Then orthogonality of the strong epimorphism $p : V \rightarrow V^c$ and the monomorphism $i : W^c \rightarrow [A, W]$ implies the existence of a unique morphism $h^c : V^c \rightarrow W^c$ such that $p.h = h^c.p$ (and consequently also $i.h^c = [A, h].i$). This defines the functor $(-)^c$ (up to an unimportant natural isomorphism).

For any object X of \mathcal{C} , the top-right path of the commutative diagram

$$\begin{array}{ccccccc} XA & \xrightarrow{\eta} & [A, AXA] & \xrightarrow{[A, c_A, XA]} & [A, XA^2] & \xrightarrow{[A, 1t_1^{-1}]} & [A, XA^2] \\ \downarrow 1e & & \downarrow [A, 11e] & & \searrow [A, 1e1] & & \downarrow [A, 1m] \\ X & \xrightarrow{\eta} & [A, AX] & \xrightarrow{[A, c]} & [A, XA] & & \end{array}$$

is the morphism (6.5) for the Hopf module XA in Proposition 6.2. Now since $e \in \mathcal{Q}$,

$$XA \xrightarrow{1e} X$$

is a regular, hence strong, epimorphism. On the other hand, $e1: AX \rightarrow X$ is also a (regular) epimorphism, so the functor $A(-): \mathcal{C} \rightarrow \mathcal{C}$ is faithful, from which it follows that

$$X \xrightarrow{\eta} [A, AX] \xrightarrow{[A,c]} [A, XA]$$

is a monomorphism; thus these must give the factorization of (6.5), and so X is indeed isomorphic to $(XA)^c$.

The natural isomorphism $V^c A \cong V$, for any Hopf module V , is provided by the morphisms n in (6.2) of Lemma 6.3. Their naturality is an obvious consequence of the naturality of the constituent morphisms. \square

Remark 6.5. If A is a Hopf monoid with unit $u: I \rightarrow A$ in a closed braided monoidal category \mathcal{C} in which idempotent morphisms split, then our assumptions in Theorem 6.4 hold with any choice of \mathcal{Q} containing the split epimorphisms.

Indeed, the multiplication m is evidently non-degenerate and the counit e is an epimorphism split by u .

Furthermore, for any Hopf module V there is an idempotent morphism

$$V \xrightarrow{1u} VA \xrightarrow{(v^1)^{-1}} VA \xrightarrow{v} V.$$

By assumption it admits a splitting $V \xrightarrow{p} V^c \xrightarrow{j} V$; and we claim that it gives rise to a splitting

$$V \xrightarrow{p} V^c \xrightarrow{j} V \xrightarrow{\eta} [A, AV] \xrightarrow{[A,c]} [A, VA] \xrightarrow{[A,v]} [A, V] \quad (6.6)$$

as required in Theorem 6.4.

Indeed, the morphism of (6.6) is equal to (6.5) by the commutativity of

$$\begin{array}{ccccccc}
V & \xrightarrow{1u} & VA & \xrightarrow{(v^1)^{-1}} & VA & \xrightarrow{v} & V \\
\eta \downarrow & & & & & & \downarrow \eta \\
[A, AV] & \xrightarrow{[A, 11u]} & [A, AVA] & \xrightarrow{[A, 1(v^1)^{-1}]} & [A, AVA] & \xrightarrow{[A, 1v]} & [A, AV] \\
[A, c] \downarrow & & \downarrow [A, c_A, VA] & & \downarrow [A, c_A, VA] & & \downarrow [A, c] \\
[A, VA] & \xrightarrow{[A, 1u1]} & [A, VA^2] & \xrightarrow{[A, (v^1)^{-1}1]} & [A, VA^2] & \xrightarrow{[A, v1]} & [A, VA] \\
& \searrow & \downarrow [A, 1m] & \text{(4.5)} & \downarrow [A, 1m] & & \downarrow [A, v] \\
& & [A, VA] & \xrightarrow{[A, (v^1)^{-1}]} & [A, VA] & \xrightarrow{[A, v]} & [A, V]
\end{array}$$

On the other hand,

$$V^c \xrightarrow{j} V \xrightarrow{\eta} [A, AV] \xrightarrow{[A,c]} [A, VA] \xrightarrow{[A,v]} [A, V]$$

is a split monomorphism since j is so and also $[A, v.c].\eta$ is so by the commutativity of

$$\begin{array}{ccccccc}
 V & \xrightarrow{\eta} & [A, AV] & \xrightarrow{[A,c]} & [A, VA] & \xrightarrow{[A,v]} & [A, V] \\
 \downarrow u1 & & & & & & \downarrow u1 \\
 AV & \xrightarrow{1\eta} & A[A, AV] & \xrightarrow{1[A,c]} & A[A, VA] & \xrightarrow{1[A,v]} & A[A, V] \\
 \parallel & \swarrow \epsilon & & & & & \downarrow \epsilon \\
 AV & & & & & & V \\
 \downarrow c & & & & & & \\
 VA & \xrightarrow{v} & & & & & V
 \end{array}$$

v

in which the left-bottom path is an identity arrow by the unitality of v ; cf. Remark 5.2. Since p is a split hence strong epimorphism, this completes the proof.

Remark 6.6. Our assumptions in Theorem 6.4 also hold in another important example, when A is a non-zero multiplier Hopf algebra [16] in the closed symmetric monoidal category of vector spaces over an arbitrary field. In this case \mathcal{Q} can be chosen to consist of all surjective linear transformations. (In this category, the epimorphisms, the strong epimorphisms, the regular epimorphisms, and the split epimorphisms all coincide: they are the surjective linear maps.)

The multiplication is non-degenerate hence non-zero by assumption. This implies that the counit e is a non-zero map to the base field; hence it is surjective.

On the other hand, in the category of vector spaces any morphism (that is, linear map) $f: X \rightarrow Y$ factorizes through the surjection $X \twoheadrightarrow \text{Im}(f)$ via the inclusion $\text{Im}(f) \hookrightarrow Y$. In the (abelian) category of vector spaces this is in turn a strong epi-mono factorization. Since vector spaces are in particular flat modules over the ground field, for any vector space X the monoidal product functor $X \otimes (-)$ preserves monomorphisms.

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